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On uniqueness of time-consistent Markov policies for quasi-hyperbolic consumers under uncertainty*

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Abstract

We give a set of sufficient conditions for uniqueness of a time-consistent Markov stationary consumption policy for a quasi-hyperbolic household under uncertainty. To the best of our knowledge, this uniqueness result is the first presented in the literature for general settings, i.e. under standard assumptions on preferences, as well as some new condition on a transition probability. This paper advocates a "generalized Bellman equation" method to overcome some predicaments of the known methods and also extends our recent existence result. Our method also works for returns unbounded from above. We provide few natural followers of optimal policy uniqueness: convergent and accurate computational algorithm, monotone comparative statics results and generalized Euler equation.

1 Introduction

The problem of dynamic inconsistency in sequential decision problems was introduced in the seminal paper of Strotz (1956), further developed in the work of Phelps and Pollak

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(1968) and Peleg and Yaari (1973), and it has has played an increasingly important role in many fields in economics (see Bernheim, Ray, and Yeltekin (2015); Chatterjee and Eyigungor (2016); Drugeon and Wigniolle (2016); Echenique, Imai, and Saito (2016); Jackson and Yariv (2014, 2015); Nakajima (2012); Sorger (2004) for some recent contributions). The classical toolkit for analyzing such "time" consistency problems was first proposed by Strotz (1956), emphasizing the language of recursive decision theory, where one searches for dynamically consistent plans in sequential optimization frameworks by imposing additional constraints which are well-known to be difficult to formulate. As observed by many researchers in subsequent discussions (e.g., Peleg and Yaari (1973) and Bernheim and Ray (1986)), the existence of such optimal dynamically consistent plan in a class of Markovian solutions is problematic, let alone the question of such solution uniqueness. Further, even when such plans do exist, they can be difficult to characterize and/or compute (see Caplin and Leahy (2006)).

As a way of circumventing these problems, Peleg and Yaari (1973) proposed a dynamic game interpretation of the time-consistency problem. More specifically, in this view of the problem, one envisions the decisionmaker as playing a dynamic game between one's current self, and each of her future "selves", with the solution concept being a subgame-perfect Nash equilibrium (SPNE, henceforth). But even, if the question of existence of SPNE is resolved, the equilibrium existence or uniqueness in the particular class of functions, namely Stationary Markov Nash Equilibria (henceforth, SMNE) is still not guaranteed (see Bernheim and Ray (1986) and Leininger (1986)).[1] Here, we also refer the reader to the interesting recent work of Maliar and Maliar (2006, 2016) who motivate why providing sharp numerical algorithms to compute (the unique) SMNE of the quasi-hyperbolic discounting decision problem can be a difficult problem to resolve.

In this paper, we seek conditions under which a simple, stable iterative numerical algorithm can be developed that (i) characterizes the existence of SMNE from a theoretical perspective, (ii) provides explicit and accurate method for computing the solution, and (iii) facilitates the characterization of monotone comparative statics. From the perspective of the existence question, our paper is very closely related to the important papers of Bernheim and Ray (1986) or Harris and Laibson (2001), where the authors add noise of invariant support in an effort to develop conditions that guarantee the existence of a time-consistent policy of locally bounded variation and/or Lipschitz for sufficiently small amount of hyperbolic discount factor. It bears mentioning, though, there is a critical difference between approaches in this literature, and those advocated in the present paper:

[1]The works of Kocherlakota (1996) and Maskin and Tirole (2001) provide an extensive set of motivations for why one might be interested in concentrating on SMNE, as opposed to SPNE.

the methods we propose do not rely on so-called "generalized Euler equations" (as, for example, in Harris and Laibson). Rather, in our paper, we propose a "generalized Bellman equation", where a new "value iteration" method is proposed. This we feel provides an important new approach when compared to this existing literature, as it allows us to link the stochastic games studied in Harris and Laibson (2001), with a recursive or value function methods suggested by Strotz (1956) (and further developed by Caplin and Leahy (2006)).

Recently Chatterjee and Eyigungor (2016) propose a method to show existence of a *continuous* randomized MSNE in a quasi-hyperbolic discounting model with a strictly positive lower bound on wealth. Specifically, they show that once consumers are allowed to randomize their investment strategies (keeping the expected investment constant) they will endogenously choose a strategy that concavifies the expected value function. Our approach is similar, but by attacking the problem for a stochastic games perspective, the conditions are placed on the primitives of the stochastic game that in essence "concavifies" the continuation expected utility exogenously. Aside from not requiring one to resort ot lotteries, this has the additional benefit relative to Chatterjee and Eyigungor (2016) as it allows to state simple sufficient conditions for obtaining *unique Lipschitzian* pure MSNE. Therefore, our new methods nicely complement those of Chatterjee and Eyigungor (2016) on the existence of continuous MSNE in such time consistency problems.

More specifically, under standard assumptions on preferences, and a new condition on a transition probability that have often been applied in the existing literature on stochastic games, we are able to develope a monotone value iteration approach to show existence and uniqueness of time-consistent policy. Further, we are able to provide sharp characterizations of their Lipschitzian structure, as well as their monotonicity properties. Finally, and equally as important, as we obtain sufficient conditions for the *uniqueness* of Markovian equilibrium optimal time-consistent policy on a minimal state space. It is worth mentioning that our methods work for returns that are bounded or unbounded above.

We are also able to construct a simple approximation scheme computing unique SMNE value in the appropriate norm, as well as conduct monotone comparative statics with the model parameters. These comparative statics and approximation results are important for applied research in the field. For example, in Sorger (2004), he proposes settings under which any twice continuously differentiable function can be supported as a policy of a time consistent hyperbolic consumer. This result can be subsequently linked to a Gong, Smith, and Zou (2007) text, showing that a hyperbolic discounting is not observationally equivalent to exponential discounting. However, the two models have *radically* *different* comparative statics. Hence, our approach allows us to sort out the exact nature of this question, and provide theoretical monotone comparative statics to clarify empirical questions that are asked by applied researchers.

2 Main results

In the environment we study, we envision an individual decisionmaker to be a sequence of "selves" indexed in discrete time $t \in \{0, 1, \ldots\}$. For a given state $s_t \in S$ (where $S = [0, \infty)$), the "self t" chooses a consumption $c_t \in [0, s_t]$, and leaves $s_t - c_t$ as an investment for future "selves". As in effect, we rule out borrowing; also we interpret the asset as a productive one, and refer to it as capital. These choices, together with current state s_t, determine a transition probability $Q(ds_{t+1}|s_t - c_t, s_t)$ of a next period state.

Self t preferences are represented by a utility function given by:

$$u(c_t) + \beta E_t \sum_{i=t+1}^{\infty} \delta^{i-t} u(c_i),$$

where $1 \geq \beta > 0$ and $1 > \delta \geq 0$, u is a instantaneous utility function and expectations E_t are taken with respect to a realization of a random variable s_i drawn each period from a transition distribution Q (see Ionescu-Tulcea theorem).

2.1 Generalized Bellman operator

Under some natural continuity assumptions on u and Q (to be specified later), we can define a Markovian equilibrium pure strategy to be an $h \in \mathcal{H}$, where $\mathcal{H} = \{h : S \to S | 0 \leq h(s) \leq s, h \text{ is Borel measurable}\}$ that is time-consistent for the quasi-hyperbolic consumer. That is, Markovian equilibrium pure strategy h satisfies the following pair functional equations:

$$h(s) \in \arg\max_{c \in [0,s]} u(c) + \beta\delta \int_S V_h(s') Q(ds'|s - c, s), \tag{1}$$

where $V_h : S \to \mathbb{R}$ is a continuation value function for the household of "future" selves following a stationary policy h from tomorrow on. The value in the Markovian equilibrium for the future selves, therefore, must solve the following additional functional equation in the continuation given as follows:

$$V_h(s) = u(h(s)) + \delta \int_S V_h(s')Q(ds'|s - h(s), s).$$

Therefore, if we define the value function for the self t to be:

$$W_h(s) := u(h(s)) + \beta\delta \int_S V_h(s')Q(ds'|s - h(s), s),$$

one obtains the relation

$$V_h(s) = \frac{1}{\beta}W_h(s) - \frac{1-\beta}{\beta}u(h(s)). \tag{2}$$

Equation (2) is our *generalized Bellman equation*. It shows a condition that any Markovian value must satisfy to solve our original maximization problem. Element $\frac{1-\beta}{\beta}u(h(s))$ is our adjustment to account for changing preferences. For $\beta = 1$ case equation (2) reduces to the standard Bellman equation. Based on equation (2), we can define an operator whose fixed points, say V^*, correspond to values for some *time-consistent* Markov policies.

2.2 Assumptions

Allowing for returns that are unbounded above (bounded below), let $(K_j)_{j\in\mathbb{N}}$ be a sequence of increasing (in a set inclusion sense) and compact subsets of S such that, each of K_j contains 0, and $S = \bigcup_{j=1}^{\infty} K_j$. For $v : S \mapsto \mathbb{R}$, such that for each $j \in \mathbb{R}$ v is bounded on K_j, and $j \in \mathbb{N}$, let us define a seminorm (see Matkowski and Nowak (2011)):

$$||v||_j := \sup_{s\in K_j} |v(s)|.$$

Put $m_j := \frac{||u||_j}{1-\beta}$ and define:

$$||v|| := \sum_{j=1}^{\infty} \frac{||v||_j}{m_j}\beta^j,$$

with a convention $||v|| = \infty$, if the series on the right hand side tends to ∞. By $M(S)$ we denote a set of real valued and Borel measurable functions on S. Consider a vector space

$$\mathcal{V} := \{v \in M(S) : v(0) = 0, \text{ and for all } j \in \mathbb{N}, ||v||_j < \infty, \text{ and } ||v|| < \infty\}$$

and denote

$$\mathcal{V}^m := \{v \in \mathcal{V} : ||v||_j \le m_j \text{ for each } j \in \mathbb{N}\}.$$

We can now state our fundamental assumption on the primitives of the stochastic game:

Assumption 1 *Let us assume:*

- $u : S \to \mathbb{R}_+$ *is continuous, increasing and strictly concave with* $u(0) = 0$.
- *for any* $s, i \in S$ $Q(\cdot|i,s) = p(\cdot|i,s) + (1 - p(\cdot|i,s))\delta_0(\cdot)$, *where* δ_0 *is a delta Dirac measure concentrated at point* 0, *while* $p(\cdot|i,s)$ *is some measure such that*
  - *for each* $s \in S \setminus \{0\}$, $i \in [0,s]$ $p(S|i,s) < 1$ *and* $p(S|0,0) = 0$*;*
  - *for each* $j \in \mathbb{N}$ $p(K_{j+1}|i,s) = p(S|i,s)$ *if* $s \in K_j$ *and* $i \in [0,s]$*;*
  - *for each* $v \in \mathcal{V}^m$, *the function*

    $$(s,i) \mapsto \int_S v(s')p(ds'|i,s)$$

    is continuous, increasing and concave with i.
- *the sequence* $(m_j)_{j\in\mathbb{N}}$ *satisfies*

  $$\delta \sup_{j\in\mathbb{N}} \left\{ \frac{m_{j+1}}{m_j} \right\} \le \beta.$$

We make a few remarks.

First, our assumptions on preferences are completely standard. That is, here, we only mention the imposition of strict concavity of a utility function in these assumptions allows us to restrict attention to single valued best replies in the equation (1); hence, we can study the fixed points of a single valued operator whose fixed points generate corresponding equilibrium values and corresponding policies in the game. It bears mentioning that a careful reading of the proof of our main existence theorem below (Theorem 1) indicates this assumption can be *weakened* if existence of MSNE is all that one seeks, as we can also work with increasing selections from a best response correspondence (not necessarily unique valued best replies).

Second, our assumption on a transition probability requires a few remarks. Q has the specific form in our conditions above. We should mention that although this is a powerful technical assumption, the conditions are satisfied in many applications (e.g., see the discussion in Chassang (2010) for a particular example of this exact structure). Additionally, as we assume positive returns (i.e., $u(\cdot) \geq 0$), our assumptions above assure that the expected continuation value is monotone in its arguments. This structure is common in the literature. For example, a stronger version of this assumption was introduced by Amir (1996), used in a series of papers by Nowak (see Balbus and Nowak (2008); Nowak (2006) and references within), as well as studied extensively in the context of games of strategic complementarities with public information in Balbus, Reffett, and Woźny (2014). We refer the reader to our related paper (see Balbus, Reffett, and Woźny (2015)) for a detailed discussion of the nature of these assumptions.

Remark 1 *Observe that we do not require that p is a probability measure. A typical example of p is: $p(\cdot|i,s) = \sum_{j=1}^{J} g_j(i,s)\eta_j(\cdot|s)$, where $\eta_j(\cdot|s)$ are measures on $\mathcal{S}$ and $g_j : S \times S \to [0,1]$ are continuous functions (increasing and concave with i) with $\sum_{j=1}^{J} g_j(\cdot) \leq 1$. However there are many examples of p that cannot be expressed by a linear combination of stochastic kernels, and still satisfy our assumptions.*

Finally, we should mention that our assumption imposes that the expected value functions stays concave. This is similar to a randomization technique advocated recently by Chatterjee and Eyigungor (2016). To see the nature of this assumption in the relation to their endogenous concavification result, observe that whenever v is not concave at the neighbourhood of zero capital then the optimal endogenous randomization would require choosing an atom at zero, exactly as required by our assumption. The difference here is that our assumption is of a global nature (i.e. satisfied for any candidate, measurable v) rather than the local one, and hence the question of pure MSNE uniqueness can be attacked.

Remark 2 *Assumption imposed on the sequence of $(m_j)_{j\in\mathbb{N}}$ is required to prove MSNE existence and uniqueness for unbounded (from above) returns. A special case of our assumption is, when u is in fact bounded on S. In fact assumptions can be even further weakened, if one works with bounded state space S.*

2.3 MSNE uniqueness

We start with noting an important auxiliary result.

Lemma 1 *$\mathcal{V}$ is a Banach space and $||\cdot||$ is its norm.*

Proof: It follows from Remark 1 and Lemma 1 in Matkowski and Nowak (2011). ■

It is easy to verify, that $\mathcal{V}^m$ is a closed subset of $\mathcal{V}$, hence by Lemma 1 a complete metric space. Following Rincon-Zapatero and Rodriguez-Palmero (2003, 2009) we define $k-$local contractions:

Definition 1 *Let* $k \in \{0,1\}$. *An operator* $T : \mathcal{V}^m \mapsto \mathcal{V}^m$ *is* k-local contraction *with modulus* $\gamma \in (0,1)$ *if for each pair* $V_1, V_2 \in \mathcal{V}^m$

$$||T(V_1) - T(V_2)||_j \leq \gamma ||V_1 - V_2||_{j+k}.$$

We now construct an operator $T : \mathcal{V} \mapsto \mathcal{V}$ by:

$$TV(s) = \frac{1}{\beta} AV(s) - \frac{1-\beta}{\beta} u(BV(s)),$$

where the pair of operators A and B defined on space $\mathcal{V}^m$ are given by:

$$\begin{aligned} AV(s) &= \max_{c \in [0,s]} \left\{ u(c) + \beta\delta \int_S V(y) Q(ds'|s-c, s) \right\}, \\ BV(s) &= \arg\max_{c \in [0,s]} \left\{ u(c) + \beta\delta \int_S V(y) Q(ds'|s-c, s) \right\}. \end{aligned}$$

Notice, in the above, we have defined the operator B to map between candidates for equilibrium values $\mathcal{V}$ to spaces of pure strategy best replies $\mathcal{H}$. So in effect, we have a pair of operator equation we need to solve to construct equilibrium values $V^* \in \mathcal{V}$. Recall also that:

$$TV(s) = u(BV(s)) + \delta \int_S V(s') Q(ds'|s - BV(s), s). \tag{3}$$

For each $j \in \mathbb{N}$ let $\mathcal{V}_j$ be a set of all restrictions of $\mathcal{V}$ to K_j. Endow, $\mathcal{V}_j$ with natural componentwise order.

Lemma 2 *Let* $j \in \mathbb{N}$, $s \in K_j$, V_1 ,$V_2 \in \mathcal{V}_j$, *and suppose that* $V_1(s') \leq V_2(s')$ *for each* $s' \in K_{j+1}$. *Then* $BV_1(s) \geq BV_2(s)$, *and* $TV_1(s) \leq TV_2(s)$.

Proof: To see monotonicity of B, consider a function $G : [0,s] \times \mathcal{V}_{j+1} \mapsto \mathbb{R}$

$$G(c, V) = u(c) + \beta\delta \int_S V(s') p(ds'|s-c, s).$$

Then for any $V \in \mathcal{V}_j$ the function $G(\cdot, s, V)$ is supermodular.

Moreover, $(c, V) \to \int_S V(s')p(ds'|s-c, s)$ has decreasing differences. To see this fact, observe we have the following inequalities:

$$\begin{aligned} 0 &\leq \int_S V_2(s')p(ds'|s-c_1, s) - \int_S V_2(s')p(ds'|s-c_2, s), \\ &= \int_S V_2(s')[p(ds'|s-c_1, s) - p(ds'|s-c_2, s)], \\ &\leq \int_S V_1(s')[p(ds'|s-c_1, s) - p(ds'|s-c_2, s)], \end{aligned}$$

where $V_2 \geq V_1$ and $c_2 \geq c_1$. Therefore, the function $(c, V) \to G(c, V)$ has decreasing differences on $[0, s] \times \mathcal{V}_{j+1}$. Since $[0, s]$ is a lattice and $\mathcal{V}_{j+1}$ is a poset, we obtain by Topkis (1978) theorem that the (unique) best reply $BR(V)(s) = \arg\max_{c \in [0,s]} G(c, V)$ is decreasing on $\mathcal{V}_{j+1}$. Since A is increasing and B decreasing, by definition of T, we conclude that T is increasing. ■

The following lemma is straightforward to prove.

Lemma 3 *For each $j \in \mathbb{N}$, $V \in \mathcal{V}$, $s \in K_j$ and constant $k \in \mathbb{N}$, we have $B(V+k)(s) = BV(s)$, and $A(V+k)(s) = AV(s) + \beta\delta k$. As a result, $T(V+k)(s) = TV(s) + \delta k$.*

Lemma 4 *T maps $\mathcal{V}^m$ into itself.*

Proof: Let $V \in \mathcal{V}^m$, $j \in \mathbb{N}$ and $s \in K_j$ be given. Observe that $BV(s) \in K_j$. Then by definition of T we have

$$TV(s) \leq (1-\beta)\, m_j + \delta \int_{K_j} V(s')Q(ds'|s - BV(s), s) \leq (1-\beta)\, m_j + \delta m_{j+1} \tag{4}$$

$$\leq (1-\beta)\, m_j + \beta m_j = m_j. \tag{5}$$

Here, (4) follows from our assumption on the sequence of $(m_j)_{j\in\mathbb{N}}$. Since j and s were fixed arbitrarily, (5) implies that $TV(\cdot) \in \mathcal{V}^m$. ■

Lemma 5 *$T : \mathcal{V}^m \mapsto \mathcal{V}^m$ is 1-local contraction with modulus δ.*

Proof: Let $V_1, V_2 \in \mathcal{V}^m$, $j \in \mathbb{N}$, $s \in K_j$, and put $k_0 := ||V_1 - V_2||_{j+1}$. Then, by Assumption 1 $Q(K_{j+1}|s - BV_i(s), s) = 1$. Then by Lemma 3,

$$T(V_i + k_0)(s) = TV_i(s) + \delta k_0.$$

Further by Lemma 2,

$$TV_2(s) - \delta k_0 = T(V_2 - k_0)(s) \leq TV_1(s) \leq T(V_2 + k_0)(s) = TV_2(s) + \delta k_0.$$

Hence, $|TV_1(s) - TV_2(s)| \leq \delta k_0$. The proof is complete, since $s \in K_j$ is chosen arbitrary. ■

For any fixed point V^* of an operator T, this value function corresponds to a stationary, time-consistent Markov policy $h^* = BV^* \in \mathcal{H}$. Equip the space of pure strategies $\mathcal{H}$ with a pointwise partial order. In this case, we obtain our main result:

Theorem 1 (Uniqueness of MSNE) *Let assumption 1 hold. Then, then there is a unique value $V^* \in \mathcal{V}^m$ and corresponding unique time-consistent Markov policy MSNE $h^* \in \mathcal{H}$. Moreover, for any $V \in \mathcal{V}^m$ we have*

$$\lim_{t \to \infty} ||T^t V - V^*|| = 0. \tag{6}$$

Proof: Observe that from Lemma 1 $(\mathcal{V}, ||\cdot||)$ is a Banach space, hence $(\mathcal{V}^m, ||\cdot||)$ is complete metric space. Furthermore, by Lemma 4 T maps $\mathcal{V}^m$ into itself, and by Lemma 5 T is 1-local contraction with modulus δ. As a result, by Theorem of Rincon-Zapatero and Rodriguez-Palmero (2003, 2009) or Matkowski and Nowak (2011) T is a contraction with respect to the metric space $(\mathcal{V}^m, ||\cdot||)$. From standard Banach Contraction Principle there is unique fixed point of T $V^* \in \mathcal{V}^m$, and (6) holds. ■

Theorem 1 is the central result of our paper. It is important for many reasons. First, it guarantees existence of time-consistent pure strategy equilibrium value V^* and policy h^*. Second, it asserts that such value and policy is unique, where the uniqueness result holds within a class of unbounded (from above) or bounded measurable value functions. This in turn implies: sequences generated by operator T are converging to V^* in the appropriate norm topology.

Such a strong characterization of time-consistent policies is obtained due to two central assumptions: concentrating on Markovian policies and the mixing assumption imposed on Q. Without these assumptions, our results would be substantially weaker. That is, the operator T is a Bellman type operator and expresses the time-consistency problem recursively for Markovian policies. However, generally if assumption 1 is not satisfied, the mapping T does not have the useful properties of similar Bellman-type operators applied in the study of optimal economies[2]. Finally, although under assumption 1 T is a contraction, the useful properties concerning equilibrium h^* characterization do not follow from standard arguments used in Stokey, Lucas, and Prescott (1989). For this reason, we present the result further characterizing the time consistent equilibrium policy functions.

Theorem 2 (Monotonicity of policies) *Assume 1, and consider a time-consistent policy h^*. If $p(\cdot|i,s)$ is constant with s, for any i, then the optimal time-consistent policy h^* is increasing and Lipschitz with modulus* 1*.*

Proof: Let $h^* = BV^*$ for $V^* = TV^*$. Consider the function

$$G(c, s, V^*) = u(c) + \beta\delta \int_S V^*(s')p(ds'|s-c).$$

Observe G is supermodular in c on a lattice $[0, s]$, and the feasible action set $[0, s]$ is increasing in the Veinott's strong set order. Moreover, by concavity of $i \to \int_S V^*(s')p(ds'|i)$ we conclude G has increasing differences with (c, s). By Topkis (1978) theorem argument maximizing h^* is increasing with s on S.

Similarly, if i denotes investment, we also can rewrite this problem as:

$$H(i, s, V^*) = u(s-i) + \beta\delta \int_S V^*(s')p(ds'|i),$$

where H is supermodular with the choice variable i on a lattice $[0, s]$, and, again, the set $[0, s]$ is increasing in the Veinott's strong set order. Again, by concavity of u, we conclude that H has increasing differences with (i, s). Therefore, again, by Topkis (1978) theorem, the optimal solution i^* is increasing with s on S.

Clearly $i^*(s) = s - h^*(s)$. Finally as both h^* and i^* are increasing on S hence h^* and i^* are Lipschitz with modulus 1. ■

Notice the results in the above theorem are also important, as they extend the result reported in Harris and Laibson (2001) on Lipschitz continuity of equilibrium to a broader scope of quasi-hyperbolic discount factors. They also provide strong structural characterizations time consistent Markov policies.

[2]It suffices to change $\delta-$Dirac measure with some other nontrivial one in assumption 1 and equilibrium uniqueness results would not hold. In such a case one could show Markov-equilibrium existence using topological arguments but with no hope of uniqueness. Also equilibrium computation would become substantially complicated (see Maliar and Maliar, 2006, 2016).

2.4 Monotone comparative statics

Next, motivated by the indeterminacy result in Gong, Smith, and Zou (2007); Maliar and Maliar (2006) (as well as concerns about the possible econometric estimation), we now consider a parameterized version of our optimization problem in the previous section of the paper. For a partially ordered set Θ , with $\theta \in \Theta$ a typical element, define the unique MSNE as h^*_θ. We make the following assumption.

Assumption 2 *Let us assume:*

- *u does not depend on θ and obeys Assumption 1.*
- *For any $s, i \in S$ and $\theta \in \Theta$ let $Q(\cdot|i,s,\theta) = (1 - p(\cdot|i,s,\theta))\delta_0(\cdot) + p(\cdot|i,s,\theta)$, where for each θ $p(\cdot|i,s,\theta)$ obeys Assumption 1;*
- *For each $V \in \mathcal{V}$ we have $(i,\theta) \to \int_S V(s')p(ds'|i,s,\theta)$ has decreasing differences with (i,θ) and $\theta \to \int_S V(s')p(ds'|i,s,\theta)$ is decreasing on Θ.*

Lemma 6 *Let $\phi : S \times \Theta \mapsto \mathbb{R}$ be a function such that $\phi(\cdot,\theta) \in \mathcal{V}$ for each $\theta \in \Theta$, and $\phi(s,\cdot)$ is decreasing for each $s \in S$. Then $\theta \mapsto T_\theta(\phi(\cdot,\theta))(s)$ is a decreasing function.*

Proof: It is easy to see that for all $V \in \mathcal{V}$, a mapping $\theta \in \Theta \mapsto A_\theta(V)$ is a decreasing function. It follows immediately from Assumption 2. We show that $B_\theta(V)$ is increasing in θ. For each $s \in S$, let us define

$$G(c,V,\theta) := u(c) + \beta\delta \int_S V(s')p(ds'|s-c,s,\theta).$$

Suppose that $c_1 < c_2 \leq s$. Then by Assumption 2, the

$$G(c_2,V,\theta) - G(c_1,V,\theta) := u(c_2) - u(c_1) + \beta\delta \int_S V(s')p(ds'|s-c_1,s,\theta) - \beta\delta \int_S V(s')p(ds'|s-c_2,s,\theta)$$

is increasing in θ. Hence by Topkis (1978) $BV_\theta(V)$ must be increasing in θ. By Assumption 2 and Lemma 2 $V \in \mathcal{V} \mapsto B_\theta(V)$ is decreasing. As a result, $B_\theta(\phi(\cdot,\theta))$ is increasing in θ. Furthermore, $\theta \mapsto T_\theta(\phi(\cdot,\theta))(s)$ is decreasing function for any $s \in S$. ■

We Assumption 2 in place, we can now prove our main result on monotone comparative statics for extremal time consistent equilibrium policies.

Theorem 3 (Monotone comparative statics) *Let Assumption 2 be satisfied. Then mapping $\theta \to h^*_\theta$ is increasing on Θ.*

Proof: Observe that by Theorem 1 $V^*_\theta(s) = \lim_{n\to\infty} T^n_\theta(\mathbf{0})(s)$, where $\mathbf{0}$ is a zero function. By Lemma 6 $T_\theta(\mathbf{0})(\cdot) \in \mathcal{V}$ and this expression is decreasing in θ. Consequently all $T^n_\theta(\mathbf{0})(s)$ satisfy all conditions of Lemma 6 and hence $V^*_\theta(s)$ decreases in θ. To finish the proof, observe that $h_\theta(\cdot) = B_\theta(V^*_\theta)(\cdot)$ and hence by Lemma 6, h_θ is increasing in θ. ■

2.5 Existence of a Generalized Euler Equation

Since Harris and Laibson (2001), many researchers have applied the so-called "generalized Euler equation" approach to solving dynamic/stochastic games. We now provide sufficient conditions for the existence of a unique differentiable MSNE, and state the version of the generalized Euler equation that characterizes MSNE that is implied by this unique MSNE.

For $V \in \mathcal{V}$, let $F_V(i) := \beta\delta \int_S V(s')Q(ds'|i)$, where $Q(\cdot|i)$ denotes transition $Q(\cdot|i,s)$ that is independent on s. We start with the following assumption.

Assumption 3 *Assume that*

- *u is twice continuously differentiable,*
- *for any a.e. differentiable V, function F_V is twice continuously differentiable on $S \setminus \{0\}$,*
- $\lim_{c\to 0} u'(c) = \infty$ *and* $\lim_{i\to 0} F'_V(i) = \infty$,
- *there exists $d > 0$ s.t. $|u''(s)| > d$ or $|F''_V(s)| > d$ for any $s \in S$.*

Clearly, the Inada type conditions are assumed to obtain interior solution while the uniform bound d on second derivatives to apply the local/global Implicit function theorem. The next remark discusses the class of stochastic transitions that satisfy our conditions.

Remark 3 *A class of measures Q satisfying assumption 3 was provided by Amir (1996), i.e. $Q(\cdot|i) = (1-g(i))\eta_1(\cdot)+g(i)\eta_2(\cdot)$ for twice continuously differentiable function $g : S \to [0,1]$ satisfying Inada condition. In particular, a class of transitions satisfying additionally assumption 1 can be a special case allowing η_1 to be a delta Dirac concentrated at point 0. Finally, Amir (1997) characterizes a class of measures Q satisfying assumption 3, if associated cdf $q(s|i)$ is twice continuously differentiable with i with integrable derivatives for any $s \in S$.*

We first prove a Lemma that shall be used in the sequel:

Lemma 7 *Assume 1 and 3 and that V' exists. Then, $BV(\cdot)$, $AV(\cdot)$, and consequently $TV(\cdot)$ is continuously differentiable on $(0,\infty)$.*

Proof: Clearly

$$AV(x) := \max_{c\in[0,x]}(u(c) + F_V(x-c)).$$

As u is strictly concave, and F is weakly concave, by Assumption 3 $BV(\cdot)$ is a unique maximizer satisfying

$$u'(BV(x)) = F_V'(x - BV(x)).$$

Equivalently $i_V(x) := x - BV(x)$ is a maximizer of $i \in [0,x] \mapsto u(x-i)+F_V(i)$. Moreover, $BV(x) \in (0,x)$ for each $x > 0$. Given our strict concavity and smoothness conditions, the local implicit function theorem imply i_V and h are locally continuously differentiable on the interior of $[0,\bar{x}]$, for each $\bar{x}$. As i_V and h is also continuous, by the global Implicit Function Theorem (see Phillips (2012), Lemma 2), we conclude that the optimal choices: h and i_V are differentiable on $(0,\infty)$. ■

Theorem 4 *Under assumption 1 and 3 MSNE policy h^* and MSNE value V^* are differentiable on $(0,\infty)$.*

Proof: By Theorem 1 $T^n(\mathbf{0}) \rightrightarrows V^*$. By Lemma 7 all $T^n(\mathbf{0})$ are differentiable on $(0,\infty)$. Denote $i^n(x) := x - B(T^n(\mathbf{0}))(x)$, it is increasing, hence $x - BV^*(x)$ is an increasing function. Then by Lebesgue Theorem (see Theorem 17.12 in Hewitt and Stromberg (1965)), there is a Lebesgue null set N, such that for $x \in S \setminus N$ i^* has a finite derivative. Hence BV^*, and consequently V^* has derivative on all $x \notin N$. Since $V^*(x) = TV^*(x)$, hence by Lemma 7 V^* is twice continuously differentiable for $x > 0$. ■

Suppose i^* is a differentiable MPNE investment, i.e. $i^*(s) := s - h^*(s)$. to simplify notation we drop $*$ from V^* and i^*. Additionally, by $q(\cdot|i)$ denote a cdf associated with measure Q (such that assumption 3 is satisfied). Similarly to Harris and Laibson (2001) or Judd (2004) we can now write the generalized Euler equations characterizing MSNE investment i:

$$u'(s - i(s)) = \beta\delta \frac{d}{di}\int_S V^*(s')dq(s'|i(s)). \tag{7}$$

Then:

$$V'(s) = u'(s - i(s))(1 - i'(s)) + \delta i'(s) \frac{d}{di} \int_S V^*(s') dq(s'|i(s)). \tag{8}$$

Using the Fundamental Theorem of the Integral Calculus for Riemann-Stieltjes integrals (see Hewitt and Stromberg (1965) Theorem 18.19 or Amir (1997), Theorem 3.2) we have:

$$\frac{d}{di} \int_S V(s') q(s'|i) = - \int_S V'(s') q'(s'|i) ds',$$

where $q'(s'|i) = \frac{d}{di} q(s'|i)$. Now integrating equation (8):

$$\begin{aligned} &\int_S V'(s) q'(s|x) ds = \\ &\int_S u'(s - i(s))(1 - i'(s)) q'(s|x) ds + \delta \int_S y'(s) \left[\frac{d}{di} \int_S V(s') dq(s'|i(s)) \right] q'(s|x) ds. \end{aligned}$$

Denote $I(x) := \frac{d}{di} \int_S V(s') dq(s'|x)$. Then:

$$-I(x) = \int_S u'(s - i(s))(1 - i'(s)) q'(s|x) ds + \delta \int_S I(i(s)) i'(s) q'(s|x) ds.$$

From equation (7):

$$-I(x) = \int_S u'(s - i(s))(1 - i'(s)) q'(s|x) ds + \frac{1}{\beta} \int_S u'(s - i(s)) i'(s) q'(s|x) ds.$$

Now, to obtain the generalized Euler equation, we can rewrite equation (7):

$$\begin{aligned} u'(x - i(x)) &= -\beta\delta \int_S u'(s - i(s))(1 - i'(s)) q'(s|i(x)) ds - \delta \int_S u'(s - i(s)) i'(s) q'(s|i(x)) ds \\ &= -\beta\delta \int_S u'(s - i(s))[1 + (\frac{1}{\beta} - 1) i'(s)] q'(s|i(x)) ds. \end{aligned}$$

The above equation is a stochastic counterpart of the Harris and Laibson (2001) or Judd (2004) generalized Euler equation. Recall, our application of Lebesgue differentiation theorem for Riemann-Stieltjes integrals is satisfied for absolutely continuous functions, a class including functions of bounded variation studied in the original construction of the generalized Euler equation by Harris and Laibson (2001). We further relate this result below when we discuss our results in relationship with the existing literature.

3 Relating the results to the literature

Equilibrium non-existence and/or multiplicity of equilibria have constituted a significant challenge for applied economists who sought to study models where dynamic consistency failures play a key role (see e.g. Maliar and Maliar, 2016). These issues have been equally as challenging for researchers that seek to identify tractable numerical approaches to computing SMNE in these (and related) dynamic games (e.g., see the discussion in Krusell and Smith (2003) or Judd (2004)).

Krusell, Kuruscu, and Smith (2002) propose a generalized Euler equation method for a version of a hyperbolic discounting consumer, and additionally obtain explicit solution for logarithmic utility and Cobb-Douglas production examples. Per the latter result, this is only an example, which is well-known to not be robust to small variations of the primitive economic data. Next Harris and Laibson (2001) and Judd (2004) proposes a generalized Euler equation approach to analyze smooth time-consistent policies and proposes a perturbation method for calculating them. The problem with this argument is providing sufficient conditions under which at any point in the state space, the generalized Euler equations represent a *sufficient* first order theory for an agent's value function in the equilibrium of the game. Concentrating on non-smooth policies, Krusell and Smith (2003) define a step function equilibrium, and show its existence and resulting indeterminacy of steady state capital levels. Further, in a deterministic setting, general existence result of optimal policies under quasi-geometric discounting can be provided using techniques proposed by Goldman (1980). The problem that this situation raises is that the multiplicity and indeterminacy of dynamic equilibrium makes using such models very difficult in applied work (for example).

To circumvent some of these mentioned predicaments authors also added noise to the decision problems or relevant dynamic games. Specifically, in a *(recursive) decision approach*, by adding noise (making payoff discountinities negligible) Caplin and Leahy (2006) prove existence of recursively optimal plan for a finite horizon decision problem and general utility functions. Similarly Bernheim and Ray (1986) show that by adding enough noise to the dynamic game (to smooth discontinuities away) existence of SMNE is guaranteed. Such *stochastic game approach* was later developed by Harris and Laibson (2001) who characterize the set of smooth SMNE by (generalized) first order conditions.

In the related paper Balbus, Reffett, and Woźny (2015) propose a similar stochastic game method for studying MSNE policies of the more general quasi-hyperbolic discounting game. Based on their generalization of the Tarski-Kantorovitch fixed point theorem they are able to show existence of the MSNE for the case of *bounded* returns in a wide range of problems, and they provide an approximate pointwise the extremal MSNE values. The question of approximating actually SMNE/MSNE that support such values remains a substantial problem in this work. In this paper, aside from relaxing conditions on the boundedness of period return functions, our generalized Bellman method provides a simple algorithm for computing both *unique* equilibrium value and *unique* MSNE.

Additionally, recently Balbus and Woźny (2016) provided an APS type method for analysing non-stationary Markovian policies of the quasi-hyperbolic discounting game numerically using set approximation techniques. One issue with this method is its inability to characterize the set of non-stationary Markovian policies that support the equilibrium value correspondence in the game.

Finally, in an interesting recent paper, Chatterjee and Eyigungor (2016) prove a existence result in randomized MSNE policies, and discuss when such equilibria exist in a class of continuous functions. As compared to our paper, note that apart from differences in assumptions (endogenous vs. exogenous concavification of the expected value function), our results differ in many important dimensions. Firstly, our existence result concerns *pure strategies*, rather then randomized policies. Second, our uniqueness result is satisfied relative to a wide class a class of bounded, measurable value functions, not just continuous values. This fact, when added with a version of our existence result proven in Balbus, Reffett, and Woźny (2015) (Theorem 5) can be used to show existence of continuous (pure) MSNE. Finally, notice our assumption on stochastic transition probability for the game requires an atom at zero asset level. This condition has a flavour of the nonexistence of a lower bound of wealth, the assumption that was shown by Chatterjee and Eyigungor (2016) to be a critical source of problems with continuous (pure) MSNE existence.

References

AMIR, R. (1996): "Strategic intergenerational bequests with stochastic convex production," *Economic Theory*, 8, 367–376.

——— (1997): "A new look at optimal growth under uncertainty," *Journal of Economic Dynamics and Control*, 22(1), 67–86.

BALBUS, Ł., AND A. S. NOWAK (2008): "Existence of perfect equilibria in a class of multigenerational stochastic games of capital accumulation," *Automatica*, 44, 1471–1479.

BALBUS, Ł., K. REFFETT, AND Ł. WOŹNY (2014): "A constructive study of Markov equilibria in stochastic games with strategic complementarities," *Journal of Economic Theory*, 150, 815–840.

——— (2015): "Time consistent Markov policies in dynamic economies with quasi-hyperbolic consumers," *International Journal of Game Theory*, 44(1), 83–112.

Balbus, Ł., and Ł. Woźny (2016): "A strategic dynamic programming method for studying short-memory equilibria of stochastic games with uncountable number of states," *Dynamic Games and Applications*, 6(2), 187–208.

Bernheim, B. D., and D. Ray (1986): "On the existence of Markov-consistent plans under production uncertainty," *Review of Economic Studies*, 53(5), 877–882.

Bernheim, B. D., D. Ray, and S. Yeltekin (2015): "Poverty and self-control," *Econometrica*, 83(5), 1877–1911.

Caplin, A., and J. Leahy (2006): "The recursive approach to time inconsistency," *Journal of Economic Theory*, 131(1), 134–156.

Chassang, S. (2010): "Fear of miscoordination and the robustness of cooperation in dynamic global games with echaxit," *Econometrica*, 78(3), 973–1006.

Chatterjee, S., and B. Eyigungor (2016): "Continuous Markov equilibria with quasi-geometric discounting," *Journal of Economic Theory*, 163, 467–494.

Drugeon, J. P., and B. Wigniolle (2016): "On time consistent policy rules for heterogeneous discounting programs," *Journal of Mathematical Economics*, 63, 174–187.

Echenique, F., T. Imai, and K. Saito (2016): "Testable implications of models of intertemporal choice: exponential discounting and its generalizations," MS.

Goldman, S. M. (1980): "Consistent plans," *Review of Economic Studies*, 47(3), 533–537.

Gong, L., W. Smith, and H.-f. Zou (2007): "Consumption and risk with hyperbolic discounting," *Economics Letters*, 96(2), 153–160.

Harris, C., and D. Laibson (2001): "Dynamic choices of hyperbolic consumers," *Econometrica*, 69(4), 935–57.

Hewitt, E., and K. Stromberg (1965): *Real and abstract analysis. A modern treatment of the theory of functions of a real variable.* Springer.

Jackson, M. O., and L. Yariv (2014): "Present bias and collective dynamic choice in the lab," *American Economic Review*, 104(12), 4184–4204.

——— (2015): "Collective dynamic choice: the necessity of time inconsistency," *American Economic Journal: Microeconomics*, 7(4), 150–178.

Judd, K. L. (2004): "Existence, uniqueness, and computational theory for time consistent equilibria: a hyperbolic discounting example," Hoover Institution, Stanford, CA.

Kocherlakota, N. R. (1996): "Reconsideration-proofness: a refinement for infinite horizon time inconsistency," *Games and Economic Behavior*, 15(1), 33–54.

Krusell, P., B. Kuruscu, and A. J. Smith (2002): "Equilibrium welfare and government policy with quasi-geometric discounting," *Journal of Economic Theory*, 105(1), 42–72.

Krusell, P., and A. Smith (2003): "Consumption–savings decisions with quasi–geometric discounting," *Econometrica*, 71(1), 365–375.

Leininger, W. (1986): "The existence of perfect equilibria in model of growth with altruism between generations," *Review of Economic Studies*, 53(3), 349–368.

Maliar, L., and S. Maliar (2006): "Indeterminacy in a log-linearized neoclassical growth model with quasigeometric discounting," *Economic Modelling*, 23(3).

——— (2016): "Ruling out multiplicity of smooth equilibria in dynamic games: a hyperbolic discounting example," *Dynamic Games and Applications*, 6(2), 243–261.

Maskin, E., and J. Tirole (2001): "Markov perfect equilibrium: I. Observable actions," *Journal of Economic Theory*, 100(2), 191–219.

Matkowski, J., and A. Nowak (2011): "On discounted dynamic programming with unbounded returns," *Economic Theory*, 46(3), 455–474.

Nakajima, M. (2012): "Rising indebtness and hyperbolic discounting: a welfare analysis," *Quantitative economics*, 3, 257–288.

Nowak, A. S. (2006): "On perfect equilibria in stochastic models of growth with intergenerational altruism," *Economic Theory*, 28, 73–83.

Peleg, B., and M. E. Yaari (1973): "On the existence of a consistent course of action when tastes are changing," *Review of Economic Studies*, 40(3), 391–401.

Phelps, E., and R. Pollak (1968): "On second best national savings and game equilibrium growth," *Review of Economic Studies*, 35, 195–199.

Phillips, P. C. B. (2012): "Folklore theorems, implicit maps, and indirect inference," *Econometrica*, 80(1), 425–454.

Rincon-Zapatero, J. P., and C. Rodriguez-Palmero (2003): "Existence and uniqueness of solutions to the Bellman equation in the unbounded case," *Econometrica*, 71(5), 1519–1555.

——— (2009): "Corrigendum to "Existence and uniqueness of solutions to the Bellman equation in the unbounded case" Econometrica, Vol. 71, No. 5 (September, 2003), 1519-1555," *Econometrica*, 77(1), 317–318.

Sorger, G. (2004): "Consistent planning under quasi-geometric discounting," *Journal of Economic Theory*, 118(1), 118–129.

Stokey, N., R. Lucas, and E. Prescott (1989): *Recursive methods in economic dynamics*. Harward University Press.

Strotz, R. H. (1956): "Myopia and inconsistency in dynamic utility maximization," *Review of Economic Studies*, 23(3), 165–180.

Topkis, D. M. (1978): "Minimazing a submodular function on a lattice," *Operations Research*, 26(2), 305–321.