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Maximum likelihood estimator for the uneven power
distribution: application to DJI returns

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Maximum Likelihood Estimator for the Uneven Power Distribution: Application to DJI Returns

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Abstract

This paper deals with estimating peaked densities over the interval $[0,1]$ using the Uneven Two-Sided Power Distribution (UTP). This distribution is the most complex of all the bounded power distributions introduced by Kotz and van Dorp (2004). The UTP maximum likelihood estimator, a result not derived by Kotz and van Dorp, is presented. The UTP is used to estimate the daily return densities of the DJI and stocks comprising this index. As the returns are found to have high kurtosis values, the UTP provides much more accurate estimations than a smooth distribution. The paper presents the program written in *Mathematica* which calculates maximum likelihood estimators for all members of the bounded power distribution family. The paper demonstrates that the UTP distribution may be extremely useful in estimating peaked densities over the interval $[0,1]$ and in studying financial data.

JEL classification: C01, C02, C13, C16, C46, C87, G10

Keywords: Density Distribution; Maximum Likelihood Estimation; Stock Returns

1. Introduction

Peaked data are more and more frequently observed in financial applications. Such data may be described using the Laplace distribution, which is an alternative to smooth distribution functions. It was only recently that Kotz and van Dorp (2004) introduced several types of two-sided power distribution defined over the interval $[0,1]$ which may serve the same purpose. However, a simple maximum likelihood estimator was present only for its basic form – the two-parametric Two-Sided Power (TP) Distribution. Every other type required a recursive optimization procedure.

Kontek (2010) has shown that such an estimator does exist for the three-parametric

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Generalized Two-Sided Power Distribution (GTP) and has demonstrated its applicability to lottery experiments. This paper shows that a similar estimator also exists for the four-parametric Uneven Two-Sided Power Distribution (UTP). Estimating densities of the DJI index and stock returns is presented as a practical application of UTP.

The lack of MLE software packages partially accounts for bounded power distributions not being more widely accepted. This paper therefore presents the program written in *Mathematica*² to calculate TP, GTP and UTP maximum likelihood estimators.

The remaining part of the paper is organized as follows. The basic UTP properties are first summarized in Point 2. The UTP maximum likelihood estimator for unimodal densities is presented in Point 3. This estimator is derived in Point 4; the estimator for non-unimodal densities is also considered. Point 5 demonstrates the use of UTP for DJI index and stock returns. Point 6 presents the *Mathematica* program. Point 7 summarizes the paper and concludes that the UTP distribution may be extremely useful in estimating peaked densities over the interval [0,1] and in researching financial markets.

2. Properties of the Uneven Two-Sided Power Distribution

2.1. The uneven two-sided power distribution UTP presented here is the Uneven Standard Two-Sided Power (USTSP) distribution as considered by Kotz and van Dorp. It has four parameters and is defined by:

$$\text{UTP}(r; \lambda, \gamma, \delta, \eta) = \begin{cases} \phi \left(\frac{r}{\lambda} \right)^{\gamma-1} & \text{if } 0 \leq r \leq \lambda, \\ \frac{\phi}{\eta} \left(\frac{1-r}{1-\lambda} \right)^{\delta-1} & \text{if } \lambda < r \leq 1. \end{cases} \quad (2.1)$$

where

$$\phi = \frac{\gamma \delta \eta}{\lambda \delta \eta + (1-\lambda) \gamma}. \quad (2.2)$$

Please note there are some minor differences between this form and the one proposed by Kotz and van Dorp. Sample shapes of the distribution are presented in Figure 2.1. One characteristic feature of UTP is its discontinuity at the point λ .

The UTP reduces to the Generalized Two-Sided Power Distribution GTP for $\eta = 1$. The function is then unimodal for $\gamma > 1$ and $\delta > 1$, J-shaped for $\gamma > 1$ and $\delta < 1$, inverse J-shaped for $\gamma < 1$ and $\delta > 1$, uniform from 0 to λ for $\gamma = 1$, uniform from λ to 1 for $\delta = 1$, U-

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shaped with anti-mode at λ for $\gamma < 1$ and $\delta < 1$, and takes the form of a power distribution for $\lambda = 1$, and a reflected power distribution for $\lambda = 0$. All shapes which are not unimodal are referred to as non-unimodal throughout this paper.

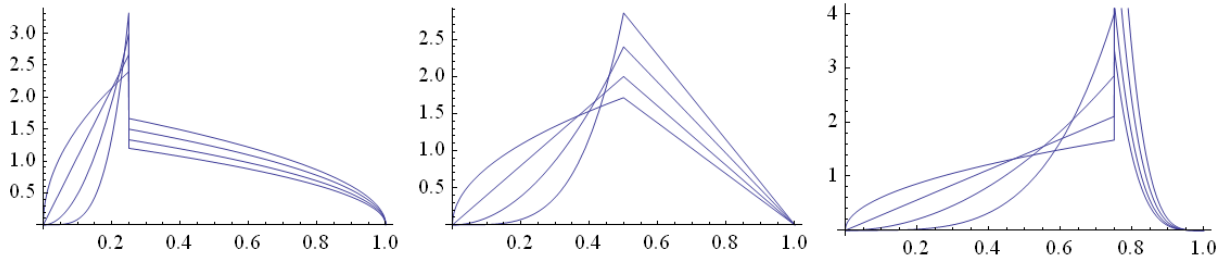


Figure 2.1. Sample shapes of the uneven two-sided power UTP distribution. On the left $\lambda = 0.25$ and $\eta = 2$, in the middle $\lambda = 0.5$ and $\eta = 1$, and on the right $\lambda = 0.75$ and $\eta = 0.5$.

A parameter of η with a value other than 1 introduces a discontinuity at λ and the shapes become more complex. However, the definition of unimodal densities as those described by $\gamma > 1$ and $\delta > 1$ is kept for convenience.

2.2. The UTP mode is given by its λ parameter:

$$\text{Mode}[\text{UTP}(r; \lambda, \gamma, \delta, \eta)] = \lambda, \quad (2.3)$$

The limit of the distribution as r approaches λ from smaller values is:

$$\lim_{r \rightarrow \lambda^-} \text{UTP}(r; \lambda, \gamma, \delta, \eta) = \phi, \quad (2.4)$$

and from larger values:

$$\lim_{r \rightarrow \lambda^+} \text{UTP}(r; \lambda, \gamma, \delta, \eta) = \frac{\phi}{\eta}. \quad (2.5)$$

It follows that η expresses the ratio of the left side to the right side limit of the distribution at point λ .

3.3. The mean value of UTP is equal to:

$$\text{Mean}[\text{UTP}(r; \lambda, \gamma, \delta, \eta)] = \frac{\gamma(\gamma(\lambda-1)(\delta\lambda+1) - \delta\lambda(\lambda(\delta\eta+\eta-1)+1) + \lambda-1)}{(\gamma+1)(\delta+1)(\gamma(\lambda-1) - \delta\eta\lambda)}. \quad (2.6)$$

3.4. The Cumulative UTP is defined as:

$$\text{CUTP}(r; \lambda, \gamma, \delta, \eta) = \begin{cases} \frac{\lambda\phi}{\gamma} \left(\frac{r}{\lambda}\right)^\gamma & \text{if } 0 \leq r \leq \lambda, \\ 1 - (1-\lambda) \frac{\phi}{\delta\eta} \left(\frac{1-r}{1-\lambda}\right)^\delta & \text{if } \lambda < r \leq 1. \end{cases} \quad (2.7)$$

which allows, after inverting, the distribution quantile to be calculated:

$$\text{Quantile}[q; \text{UTP}(p; \lambda, \gamma, \delta, \eta)] = \begin{cases} \lambda \left(\frac{q\gamma}{\lambda\phi} \right)^{\frac{1}{\gamma}} & \text{if } 0 \leq q \leq \frac{\lambda\phi}{\gamma}, \\ 1 - (1 - \lambda) \left[\frac{(1 - q)\delta\eta}{(1 - \lambda)\phi} \right]^{\frac{1}{\delta}} & \text{if } \frac{\lambda\phi}{\gamma} < q \leq 1. \end{cases} \quad (2.8)$$

In the special case where $q = 0.5$, the median is given by:

$$\text{Median}[\text{UTP}(r; \lambda, \gamma, \delta, \eta)] = \begin{cases} \lambda \left(\frac{\gamma}{2\lambda\phi} \right)^{\frac{1}{\gamma}} & \text{if } \frac{\lambda\phi}{\gamma} \geq \frac{1}{2}, \\ 1 - (1 - \lambda) \left[\frac{\delta\eta}{2(1 - \lambda)\phi} \right]^{\frac{1}{\delta}} & \text{if } \frac{\lambda\phi}{\gamma} < \frac{1}{2}. \end{cases} \quad (2.9)$$

3. Maximum Likelihood Estimator for the UTP Distribution

3.1. Although Kotz and van Dorp provided a maximum likelihood estimator for the Two-Sided Power TP distribution, they did not present a similar solution for either the Generalized Two-Sided Power GTP or Uneven Two Sided Power UTP distributions. Instead, they proposed a recursive maximum likelihood procedure for both distributions. However, Kontek (2010) has shown that such a solution does exist for GTP. It appears that similar solution exists for UTP as well. As with TP and GTP, the only restriction is that this simple and straightforward estimator only concerns unimodal densities, which in any case, seem to be the only interesting ones in most practical applications. The solution for non-unimodal densities is also present but requires solving a nonlinear equation, and more detailed considerations.

At this point, the maximum likelihood estimator for UTP in the case of unimodal densities is demonstrated without giving any details of how it was derived. This is provided in Point 4, together with the non-unimodal density analysis.

3.2. An important feature of the UTP likelihood function (similarly to TP and GTP) is that it may have multiple maxima. That these local maxima only appear at the sample points assists in finding the global maximum. Importantly, the likelihood values at these points can be calculated quite simply by using a formula and do not require any optimization algorithm. This produces a very different approach than that commonly used for smooth distributions.

The estimation procedure is based on checking the values of the likelihood function at the sample points and selecting the greatest value. Knowing the point at which the likelihood function achieves its maximum allows the sought parameters to be derived, once again, by using a formula without recourse to any optimization algorithm.

3.3. The estimator of the λ parameter is:

$$\hat{\lambda} = r_k, \quad (3.1)$$

where r_k denotes the value of the k th point from the ordered sample at which the log-likelihood function defined as:

$$\text{Log}L_k = -s(1 + \ln s) + w_k + 2k \ln k + 2m \ln m - k \ln(r_k w_k^-) - m \ln[(1 - r_k) w_k^+], \quad (3.2)$$

achieves its maximum. In (3.2), s denotes the sample count, $m = s - k$ and

$$w_k = w_k^- + w_k^+, \quad (3.3)$$

$$w_k^- = -\ln \left[\frac{\prod_{i=1}^k r_i}{r_k^k} \right], \quad (3.4)$$

$$w_k^+ = -\ln \left[\frac{\prod_{i=k+1}^s (1 - r_i)}{(1 - r_k)^{s-k}} \right], \quad (3.5)$$

where r_i denotes the value of the i th point from the ordered sample. Once the point k is determined, the maximum likelihood estimator of γ is given by:

$$\hat{\gamma} = \frac{k}{w_k^-}, \quad (3.6)$$

$$\hat{\delta} = \frac{m}{w_k^+}, \quad (3.7)$$

$$\hat{\eta} = \frac{k^2 w_k^+ (1 - r_k)}{m^2 w_k^- r_k}, \quad (3.8)$$

where w_k is calculated at point k . As mentioned, this UTP result was not provided by Kotz and van Dorp.

4. Derivation of the UTP Maximum Likelihood Estimator

4.1. Point 4 is devoted solely to the derivation of the UTP maximum likelihood estimator together with an analysis of non-unimodal densities. This is quite a technical and detailed subject and may therefore be skipped by readers more interested in practical estimation results.

4.2. Let us consider the likelihood function in the interval between points k and $k + 1$

from the ordered sample. According to (2.1) and (2.2), this may be expressed as:

$$L_k = \left(\frac{\gamma \delta \eta}{\lambda \delta \eta + (1-\lambda)\gamma} \right)^k \left(\frac{P_k^-}{\lambda^k} \right)^{\gamma-1} \left(\frac{\gamma \delta}{\lambda \delta \eta + (1-\lambda)\gamma} \right)^m \left(\frac{P_k^+}{(1-\lambda)^m} \right)^{\delta-1}, \quad (4.1)$$

where $P_k^- = \prod_{i=1}^k r_i$, and $P_k^+ = \prod_{i=k+1}^s (1-r_i)$. The interval between the lower bound of the distribution (i.e. the value 0) and the first sample point may be also considered. In this case, $k = 0$,

$P_0^- = 1$, and $P_0^+ = \prod_{i=1}^s (1-r_i)$. Similarly, for the interval between the last sample point and the

upper bound of the distribution (i.e. the value 1), $k = s$, $P_s^- = \prod_{i=1}^s r_i$, and $P_s^+ = 1$. Taking the

logarithm of (4.1) results in the log-likelihood function:

$$\begin{aligned} \text{Log}L_k &= k \left[\ln \gamma + \ln \delta + \ln \eta - \ln (\gamma - \lambda \gamma + \lambda \delta \eta) \right] \\ &\quad + m \left[\ln \gamma + \ln \delta - \ln (\gamma - \lambda \gamma + \lambda \delta \eta) \right] \\ &\quad + (\gamma - 1) (\ln P_k^- - k \ln \lambda) \\ &\quad + (\delta - 1) \left[\ln P_k^+ - m \ln (1 - \lambda) \right] \end{aligned} \quad (4.2)$$

The second derivative of (4.2) with respect to λ is:

$$\frac{d^2 \text{Log}L_k}{d\lambda^2} = \frac{k(\gamma-1)}{\lambda^2} + \frac{m(\delta-1)}{(1-\lambda)^2} + \frac{(k+m)(\gamma-\delta\eta)^2}{(\gamma-\lambda\gamma+\lambda\delta\eta)^2}, \quad (4.3)$$

which is always positive for $\gamma > 1$ and $\delta > 1$. It follows that, in the case of unimodal densities, the log-likelihood function is always convex between the points k and $k+1$ and reaches its maximum value at one of these points. Moving from one interval to another, however, changes the log-likelihood function (4.2) as it is dependent on k . The corollary of this is that the log-likelihood function is not differentiable at the sample points, and local maxima may appear there. It is therefore sufficient to check the values of the log-likelihood function at the sample points to find its global maximum. This can be done as follows.

4.3. The maximum value of the log-likelihood function at point k can be found by determining its first derivative with respect to γ , δ , and η , and comparing all the results to 0:

$$\frac{d \text{Log}L_k}{d\gamma} = \frac{s \delta \eta r_k}{\gamma [\delta \eta r_k + \gamma (1-r_k)]} - w_k^- = 0, \quad (4.4)$$

$$\frac{d \text{Log}L_k}{d\delta} = \frac{s \gamma (1-r_k)}{\delta [\delta \eta r_k + \gamma (1-r_k)]} - w_k^+ = 0, \quad (4.5)$$

$$\frac{d \text{Log}L_k}{d\eta} = \frac{k\gamma(1-r_k) + m\delta\eta r_k}{\eta[\delta\eta r_k + \gamma(1-r_k)]} = 0, \quad (4.6)$$

where λ has been substituted with r_k , and where (3.3), (3.4) and (3.5) were used to simplify the results. Solving the set of equations (4.4), (4.5) and (4.6) results in maximum likelihood estimators of γ (3.6), δ (3.7), and η (3.8). Substituting the estimated values into the log-likelihood function (4.2) results, after rearranging, in the form (3.2) which does not depend on the distribution parameters and is used for maximization. This ends the derivation of the UTP maximum likelihood estimator for unimodal densities.

4.4. It is interesting (albeit from a theoretical rather than a practical point of view) to determine the maximum likelihood estimator for non-unimodal densities. In this case, the second derivative of the log-likelihood function (4.3) may assume a negative value. This would indicate that the log-likelihood function (4.2) is concave and its maximum may be located between points k and $k + 1$.

4.4.1. In order to find a maximum, the first derivative of the log-likelihood function (4.2) with respect to λ is compared to 0:

$$\frac{d \text{Log}L_k}{d\lambda} = \frac{m(\delta-1)}{1-\lambda} + \frac{k(1-\gamma)}{\lambda} + \frac{(k+m)(\gamma-\delta\eta)}{\gamma(1-\lambda) + \delta\eta\lambda} = 0, \quad (4.7)$$

Determining whether the obtained value(s) of λ is (are) located within the considered interval $[k, k + 1]$ is not possible, because γ , δ and η are not known. One possibility is therefore to maximize the log-likelihood function (4.2) and to check the obtained value(s) of λ . Such a procedure, however, is not so convenient because (4.2) has 4 parameters.

4.4.2. The task can be simplified as follows. First, we define w_k , w_k^- , and w_k^+ more generally than (3.3), (3.4), and (3.5) as functions of λ within the interval $[k, k + 1]$ rather than the values at point k :

$$w(\lambda)_k = w(\lambda)_k^- + w(\lambda)_k^+, \quad (4.8)$$

$$w(\lambda)_k^- = -\ln \left[\frac{\prod_{i=1}^k r_i}{\lambda^k} \right], \quad (4.9)$$

$$w(\lambda)_k^+ = -\ln \left[\frac{\prod_{i=k+1}^s (1-r_i)}{(1-\lambda)^{s-k}} \right]. \quad (4.10)$$

Clearly (4.8) - (4.10) result in the same value as (3.3) - (3.5) for $\lambda = r_k$, i.e. at the bound of the considered interval $[k, k + 1]$. Calculating the first derivatives of the log-likelihood function (4.2) with respect to γ , δ and η results in:

$$\frac{d \text{Log}L_k}{d\gamma} = \frac{s \delta \eta \lambda}{\gamma [\delta \eta \lambda + \gamma (1 - \lambda)]} - w(\lambda)_k^- = 0, \quad (4.11)$$

$$\frac{d \text{Log}L_k}{d\delta} = \frac{s \gamma (1 - \lambda)}{\delta [\delta \eta \lambda + \gamma (1 - \lambda)]} - w(\lambda)_k^+ = 0, \quad (4.12)$$

$$\frac{d \text{Log}L_k}{d\eta} = \frac{k \gamma (1 - \lambda) + m \delta \eta \lambda}{\eta [\delta \eta \lambda + \gamma (1 - \lambda)]} = 0, \quad (4.13)$$

where (4.8), (4.9) and (4.10) are used to simplify the results (cf. (3.3) - (3.5)). Solving the set of equations (4.11) - (4.13) yields estimators of γ , δ , and η , as functions of λ (cf. (3.6) - (3.8))

$$\hat{\gamma}(\lambda) = \frac{k}{w(\lambda)_k^-}, \quad (4.14)$$

$$\hat{\delta}(\lambda) = \frac{m}{w(\lambda)_k^+}, \quad (4.15)$$

$$\hat{\eta}(\lambda) = \frac{k^2 w(\lambda)_k^+ (1 - \lambda)}{m^2 w(\lambda)_k^- \lambda}, \quad (4.16)$$

Substituting (4.14) - (4.16) to (4.2) yields the log-likelihood function (cf. (3.2)):

$$\begin{aligned} \text{Log}L(\lambda)_k = & -s(1 + \ln s) + w(\lambda)_k + 2k \ln k + 2m \ln m \\ & - k \ln [\lambda w(\lambda)_k^-] - m \ln [(1 - \lambda) w(\lambda)_k^+], \end{aligned} \quad (4.17)$$

which is a function of only one parameter λ and is therefore much easier to maximize than (4.2). As maximizing a function is equivalent to comparing its first derivative to 0, the presented way of proceeding may be further investigated.

4.4.3. Returning to the full form of the log-likelihood function by substituting (4.8) - (4.10) to (4.17), calculating its first derivative with respect to λ , then putting back (4.8) - (4.10) in order to simplify the result, and comparing it to 0 yields:

$$\frac{d \text{Log}L_k}{d\lambda} = -\frac{k^2}{\lambda w(\lambda)_k^-} + \frac{m^2}{w(\lambda)_k^+ - \lambda w(\lambda)_k^+} = 0, \quad (4.18)$$

Solving (4.18) with respect to λ and leaving all the $w(\lambda)$ expressions on the right-hand side gives the following nonlinear equation:

$$\lambda = \frac{k^2 w(\lambda)_k^+}{m^2 w(\lambda)_k^- + k^2 w(\lambda)_k^+}, \quad (4.19)$$

which can be solved numerically with respect to λ . The equation may have 0, 1 or more solutions.

4.4.4. The maximum of the log-likelihood function (4.17) in the considered interval $[k, k + 1]$ may be any of the λ values determined by solving (4.19). The whole maximum likelihood procedure for non-unimodal densities would therefore require the following steps:

- Solve (4.19) numerically with respect to λ
- Check whether any of the resulting λ values are located within the interval $[k, k + 1]$
- Calculate γ , δ and η for the λ value(s) using (4.14), (4.15), and (4.16)
- Calculate the second derivative of the log-likelihood function using (4.3) and check whether this value is negative (a positive value would indicate a minimum of the log-likelihood function)
- If the above conditions are satisfied, then calculate the value of the log-likelihood function using (4.17). If any one of them is not satisfied, then the maximum is either at point k or $k + 1$ as is the case with unimodal densities.
- Calculate the log-likelihood value at the bounds of the interval using (4.17) or (3.2).
- Chose the greatest value of the log-likelihood function.

The described procedure must be repeated for all intervals $[k, k + 1]$ in order to find the global maximum.

4.4.5. The procedure described in 4.4.4. may be modified by maximizing (4.17) instead of solving (4.19). Both procedures should be equally efficient.

4.5. As presented in this Point, the UTP maximum likelihood estimator is fairly easy to calculate for unimodal densities. The procedure for non-unimodal densities is not so straightforward but can be easily implemented by a software package.

5. Estimation of DJI Return Densities

5.1. Here, we consider the DJI and the stocks comprising it. The data consist of daily prices in the period from April 1, 2008 to March 31, 2010. This makes a sample of 505 values. All prices P_i have been transposed to log-returns r_i in the interval $[0,1]$ using:

$$\logret_i = \ln \frac{P_i}{P_{i-1}}, \quad (5.1)$$

$$min = (1 + \sigma) Min(logret), \quad (5.2)$$

$$max = (1 + \sigma) Max(logret), \quad (5.3)$$

$$r_i = \frac{logret_i - min}{max - min}. \quad (5.4)$$

The constant $\sigma > 0$ was introduced³ in order to avoid any resulting r_i values assuming a value of 0 or 1. This σ constant was arbitrarily set to 0.05.

5.2. Figure 5.1 presents the estimation results for the DJI index.

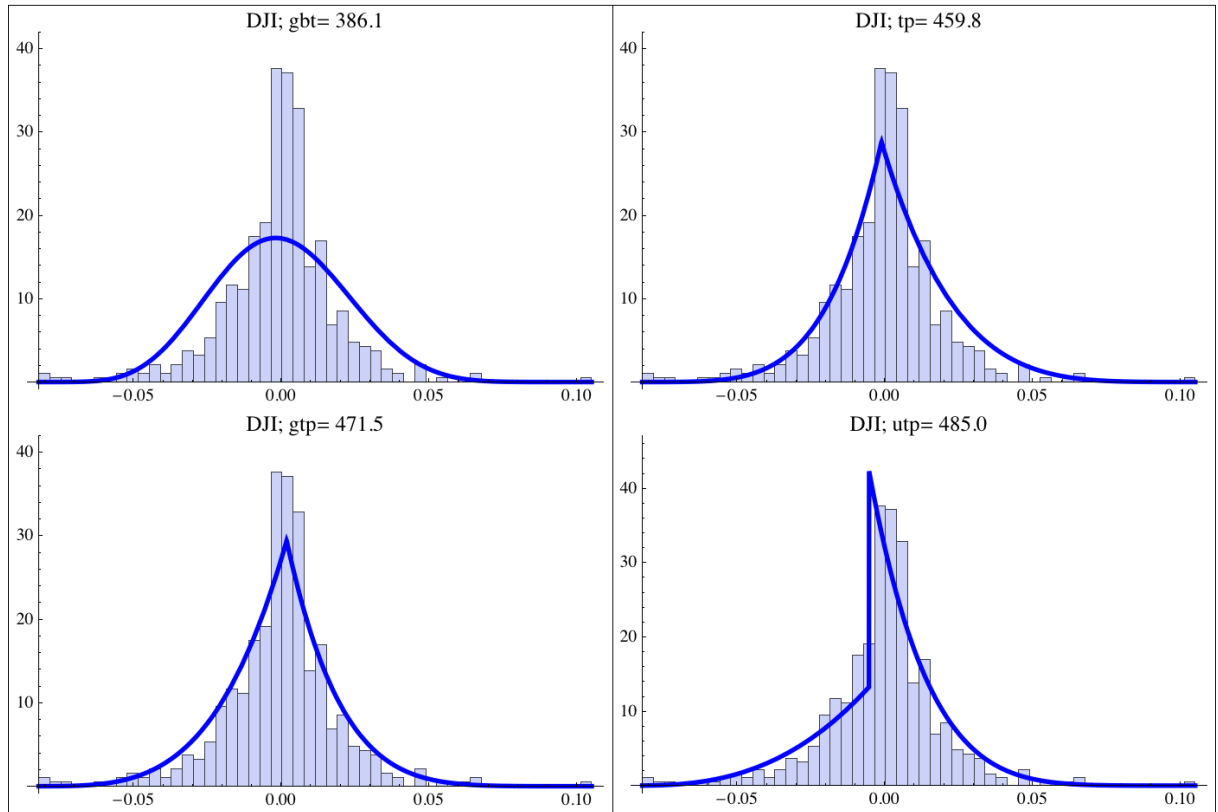


Figure 5.1. Density estimations of the DJI index daily returns. The *gbt*, *tp*, *gtp* and *utp* values indicate the maximized log-likelihood values using the respective distributions.

Four estimations are compared. The first uses the Generalized Beta Distribution GBT of Libby and Novick (1982). This is a smooth function. The next are estimations using a family of bounded power distributions: the Two-Sided Power TP, the Generalized Two-Sided Power GTP and the Uneven Two-Sided Power UTP. It is evident that power distributions provide much better estimates of DJI index returns densities than does the smooth distribution. This is indicated by the maximized log-likelihood values. This is not surprising as the kurtosis of the examined sample is 8.12, i.e. the sample is much more peaked than the normal distribution (which has a kurtosis of 3).

³ Equations (5.2) and (5.3) assume that *min* is negative and *max* is positive.

It is interesting to note that the UTP distribution has a highly asymmetric shape despite the skewness of the examined sample only having a small positive value (0.093). This is because it is skewness *together* with kurtosis and the location of the density mode on the $[0,1]$ interval, which affects the symmetry of the resulting density function. The GTP has a more “symmetric” and “nicer” shape, but offers a lower maximized log-likelihood value.

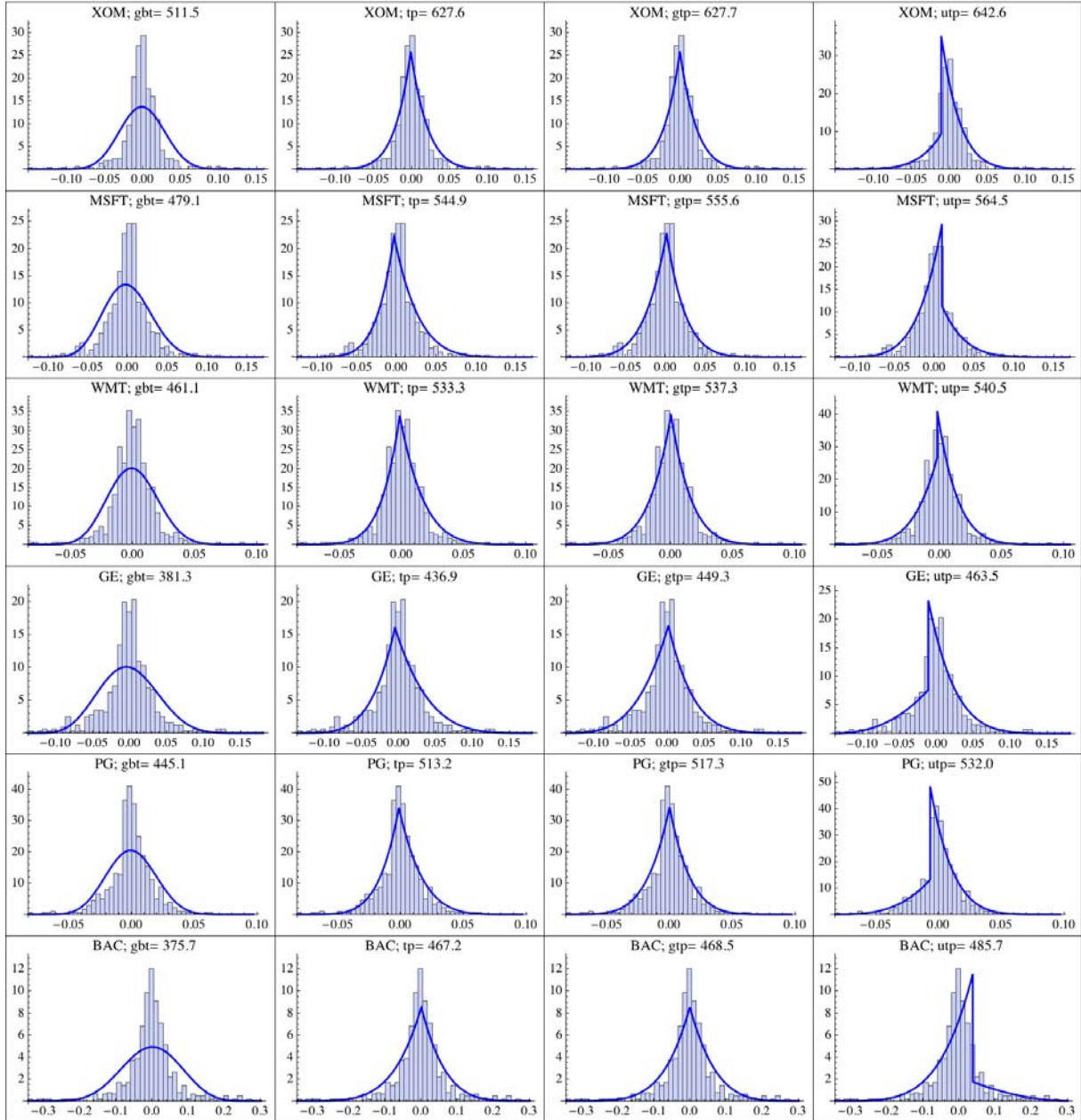


Figure 5.2. Daily return density estimations for 6 biggest companies from the DJI index. The *gbt*, *tp*, *gtp* and *utp* values indicate the maximized log-likelihood values.

5.3. The six biggest companies from the DJI index are analyzed next. The estimation results are presented in Figure 5.2. The patterns observed in Figure 5.2. are generally the same as in Figure 5.1. Power distributions perform much better than the smooth distribution in all cases. This results from the nature of stock return distributions. Table 5.1 presents distribution

moment values for the considered stocks. The very high values of kurtosis confirm the distributions are much more peaked than the normal distribution.

	XOM	MSFT	WMT	GE	PG	BAC
Mean	-0.0004	0.0001	0.0002	-0.0012	-0.0001	-0.0014
Skewness	0.246	0.307	0.214	0.095	-0.111	-0.126
Kurtosis	12.92	8.72	9.25	6.49	7.41	8.08

Table 5.1. Distribution moments for the 6 biggest companies from the DJI index.

5.4. The obtained results are interesting in the sense that the returns distributions of the DJI and the stocks comprising this index are not smooth at all. This contradicts the claim that stock returns resemble a normal distribution and provides a strong argument for using peaked distributions to describe them.

6. MLE Software

6.1. The maximum likelihood estimation procedure described in Point 3.3 may easily be packaged in a software program. The program presented in Appendix 1 is written in *Mathematica* and calculates estimations for all the bounded power distributions: TP, GTP and UTP. The maximum likelihood estimation procedures for the first two distributions, which are used by the program, were presented in another paper by Kontek (2010).

6.2. The program presented only considers unimodal densities. When γ or δ assumes a value less than 1, the solution is excluded from further analysis. This is a sound approach when the researcher has reason to believe that the sought distribution is unimodal. Solutions where either γ or δ assume a value of infinity are also excluded. This is a corollary of w_k^- or w_k^+ assuming a value of 0 (cf. (3.6) - (3.8)) (see Kontek, 2010).

6.3. One big advantage of the presented program is that it is extremely fast compared to standard optimization procedures for smooth densities. This is due to the estimation algorithm presented in this paper and due to performing calculations on the whole list of arguments, which is a feature of the *Mathematica* language.

It takes only 0.05 s to estimate a four-parametric UTP density function for a sample of 500 values. This time can be shortened even further as the algorithm is scalable and the computation can easily be parallelized on several processors. Classical optimization algorithms, by comparison, take at least 2 orders of magnitude longer to estimate the smooth GBT distribution and cannot be parallelized.

The computation time can be even further reduced whenever there is a large sample whose data assume a limited number of values. This is because it is not necessary to calculate the values of the log-likelihood function for all the sample points, but only for distinct points. In financial applications, however, the sample values are usually all different and any attempt to reduce the number of points will only lengthen the procedure.

6.4. The parameters of the presented program are the list of sample values and the number of distribution parameters *parnum*. The *parnum* equals 2 for TP, 3 for GTP, and 4 for UTP. Other comments which may help to understand the program flow are given inside the procedure. An explanation of the *Mathematica* language may be found in several textbooks (see Ruskeepää, 2009 as a good example).

7. Conclusions

This paper presented estimation procedures for peaked densities over the interval $[0,1]$. The Uneven Two-Sided Power distribution, UTP (Kotz and van Dorp, 2004) was considered for this purpose. The paper presented the UTP maximum likelihood estimator, which was not derived by Kotz and van Dorp. The UTP was used to estimate the daily return densities of the DJI and stocks comprising this index. As examined returns are found to have high kurtosis values, the UTP provides much more accurate estimations than a smooth distribution. The paper demonstrates that UTP distribution may be extremely useful in estimating peaked densities over the interval $[0,1]$ and in researching financial markets.

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Appendix 1 - The *Mathematica* program⁴ to calculate MLEs of TP, GTP, and UTP.

```

tpestimation[list_, parnum_] := (* parnum: 2 -> TP, 3 -> GTP, 4 -> UTP *)
Module[{s = Length[list], (* data count *)
  olist = Sort[list], (* ordered sample *)
  w, ws, logL, logLs, best},
w = Table[ (* {rk, wk-, wk+, k, m} for all k *)
  {Take[olist, {k}][[1]], Chop[-Total[Log[Take[olist, k]
    Take[olist, {k}][[1]]]]],
  Chop[-Total[Log[1 - Take[olist, -(s - k + 1)]
    1 - Take[olist, {k}][[1]]]]], k, s - k}, {k, s}
];
ws = Select[w, Which[ (* reject cases when wk-=0 and/or wk+=0 *)
  parnum == 2, #[2] > 0 || #[3] > 0 &, parnum >= 3,
  #[2] > 0 && #[3] > 0 &]];
logL = Switch[parnum, (* {rk, LogLk, γk, δk, ηk} for all k *)
  2, (* TP *)
  {#[1], #[2] + #[3] - s + s Log[ $\frac{s}{#[2] + #[3]}$ ]},  $\frac{s}{#[2] + #[3]}$ } &/@ ws,
  3, (* GTP *)
  {#[1], #[2] + #[3] + s (-1 - 2 Log[ $\sqrt{#[1] #[2]} + \sqrt{(1 - #[1]) #[3]}$ ] + Log[s]),
   $\frac{s}{#[2] + \sqrt{\frac{(1 - #[1]) #[2] #[3]}{#[1]}}}$ ,  $\frac{s}{#[3] + \sqrt{\frac{#[1] #[2] #[3]}{1 - #[1]}}}}$ } &/@ ws,
  4, (* UTP *)
  {#[1], -s (1 + Log[s]) + #[2] + #[3] + 2 #[4] Log[#[4]] + 2 #[5] Log[#[5]]
  - #[4] Log[#[1] #[2]] - #[5] Log[(1 - #[1]) #[3]],
   $\frac{#[4]}{#[2]}$ ,  $\frac{#[5]}{#[3]}$ ,  $\frac{#[4]^2 (1 - #[1]) #[3]}{#[5]^2 #[1] #[2]}$ } &/@ ws
];
logLs = Select[logL, Which[ (* only unimodal densities *)
  parnum == 2, #[3] > 1 &, parnum >= 3,
  #[3] > 1 && #[4] > 1 &]];
best = First[SortBy[logLs, -#[2] &]]; (* the best solution *)
{best[2], Join[{λ → best[1], γ → best[3]},
  If[parnum >= 3, {δ → best[4]}, {}], If[parnum == 4, {η → best[5]}, {}]}]}
]

```

⁴ The use of this program is free for research. The user is only asked to notify the author (i.e. Krzysztof Kontek), the affiliation (i.e. Artal Investments) and the source (i.e. this paper), also when the program is modified or re-written using another programming language (“Based on...”). The author takes no responsibility for any results obtained using the program.