Warsaw School of Economics
Institute of Econometrics
Department of Applied Econometrics

Department of Applied Econometrics
Working Papers Series
Warsaw School of Economics
Al. Niepodleglosci 164, 02-554 Warszawa, Poland

Working Paper No. 3-05

Money and prices in the Polish economy. Seasonal cointegration approach

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Paper presented to the 32nd Conference *Macromodels'2005*, Kliczków, December 2005

This paper is available at the Warsaw School of Economics
Department of Applied Econometrics website at: http://www.sgh.waw.pl/instytuty/zes/wp/

Money and prices in the Polish economy. Seasonal cointegration approach

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Abstract

The paper presents the analysis of the long run causality behaviour between money and prices in the Polish economy during the transition period. The study makes use of the monetary inflation model known as the *P-star* model, originally developed by the FED economists at the end of 80-ties. The research on the relationship between money and prices in the Polish economy carried out to date indicates that some variables (GDP, prices) show the irregular seasonal pattern. For this reason we propose to analyse the long run relationship between money and prices in the Polish economy by means of seasonal cointegration, developed by Hylleberg, Engle, Granger and You in the beginning of 90-ties.

The main hypothesis has been verified positively. The results of the research give the evidence that there exists a long-run causality relationship between money and prices (long-run cointegration relationship), which follows the assumptions of the *P-star* inflation model. The results also indicate that there are no seasonal cointegrating relationships in the *P-star* inflation model, which can be interpreted as the money demand equations. This means that the quality of the inflation forecasts cannot be improved by applying the additional seasonal cointegrating relationships to this model.

Keywords: money, inflation, P-star, money demand, real money gap, seasonality, seasonal cointegration

JEL codes: C32, E31, E41

1. Introduction

The research on the inflationary processes in many countries reveals a close relationship between money and prices. However the detailed analysis of data conducted for many years indicates that such a relationship exists only in the long–run. In the medium and short run the impact of changes in taxes, the exchange rate, foreign supply shocks or the pressure of domestic wages implicate that the actual level of prices may stand out from the equilibrium value calculated on the basis of monetary aggregates. It is important to emphasise that if the level of prices is linked to the money stock in the long-run, it may facilitate the central bank to conduct a more effective monetary policy. This means that the money holdings may be the significant indicator of the monetary policy restrictiveness and may support forecasting of the future development of inflationary processes.

The purpose of this article is to investigate whether the similar long-run relationship between money and prices existed in the Polish economy in transition period, i.e. in years 1994–2003. Regarding the fact that the literature of the money and prices relationships is very comprehensive we have chosen a model, which treats money-prices relationship as a long-term phenomenon. This model has been called the *P-star* model.

Considering the strong seasonal pattern of certain time series used in the research we proposed a seasonal cointegration approach to verify the hypothesis about existence of the money-prices relationship. The seasonal cointegration allows for both long-run equilibrium relationships between variables and common quarterly as well as biannual fluctuations.

2. Theoretical foundations of the *P-star* model

2.1. *P-star* concept

The concept of the *P-star* models has its origins in the Quantity Theory of Money and the Equation of Exchange proposed by I. Fisher (1911) is the starting point for the further consideration

$$M \cdot V = P \cdot Y, \tag{1}$$

where M denotes nominal money stock, V velocity of money, P level of prices and Y gross domestic product. Assuming that velocity is constant in the long-run or depends on other variables like GDP or interest rates we may derive direct relationship between money holdings and average price level. In that case we may calculate a theoretical price level which should be valid if no short-run distortions occurred. In the seminal paper of Hallman *et al.* (1991) this prices level has been denoted as P^* and has been defined as follows

$$P^* = \frac{M \cdot V^*}{Y^*}, \tag{2}$$

where V^* is a long-run equilibrium velocity of money and Y^* stands for potential output. Hallman *et al.* (1991) define P^* as the money stock per unit of potential output under the assumption that velocity of money is at its equilibrium level. In other words P^* means theoretical price level which would occur in the economy if the goods market and the money market would have been in equilibrium. It is noteworthy that P^* is an increasing function of the actual money holdings.

Equations (1) and (2) indicate that the short term fluctuations around the equilibrium correspond to the deviation of the actual output from the potential output or to the deviation of the actual velocity of money from its long-run value (what implies the temporary distortion of money demand). The above described relations may be illustrated (after substitution of M to (2) from (1) and rearrangement) as

$$p - p^* = (y^* - y) + (v - v^*), \tag{3}$$

where small letters denote logarithms.

Equation (3) has been called the *P-star* model. The mechanism of the model is straightforward. If the actual price level p is below its theoretical level p^* being the function of the money stock it corresponds to an increase of the real GDP above the potential output or to the fall of money velocity v below its long-run value v^*.

In the first case the return to the equilibrium level occur through the labour market by the growth of nominal wages and in the final effect by the growth of prices.

In the second case the adjustment takes place on the money market by the fall of cash holdings followed by an increase of money velocity and the growth of prices (Deutsche Bundesbank, 1992).

These properties of the *P-star* model cause that p^* may be treated as index of the inflationary pressure. If the actual price level p and the derived theoretical price level p^* move together in the long-run the difference between these variables may be a predictor for the future inflation (Hallman *et al.*, 1991, Gerlach and Svensson, 2003). The difference between the actual price level p and its theoretical equilibrium level p^*, which occurs if both output and money velocity are equal to its long-run values, has been called the **price gap** (Hallman *et al.*, 1991).

2.2. Real money gap

The price gap expressed by (3) may be written in a different form. Svensson (2000) proposes to write the left hand side of (3) for the period t and then subtract and add the current money holdings m what gives

$$p_t - p_t^* = - m_t + p_t + m_t - p_t^* = -(\tilde{m}_t - \tilde{m}_t^*), \tag{4}$$

where $\tilde{m}_t = m_t - p_t$ denotes the actual real money, and $\tilde{m}_t^* = m_t - p_t^*$ the long-run equilibrium value of the real money. After that the equation (3) may be written as

$$\tilde{m}_t - \tilde{m}_t^* = (y_t - y_t^*) + (v_t^* - v_t), \tag{5}$$

From the economic point of view the long-run equilibrium value of the real money means a hypothetical level of the real money, which would exist if actual output would be equal to the potential one and the money velocity would be equal to its long-run level.

The difference between actual and long-run levels of real money holdings ($\tilde{m}_t - \tilde{m}_t^*$) has been called by Svensson (2000) the **real money gap**[1]. It is noteworthy that the real money gap is the negative of the price gap.

The long-run value of the real money $\tilde{m}_t^*$ may be derived using the long-run function of the money demand with regards to the potential output and the interest rates, which for the Polish economy in the transition period may have a form

$$m_t^D - p_t = \tilde{m}_t^D = v_0 + \beta_1 y_t + \beta_2 (R_t^L - R_t^S). \tag{6}$$

where t denotes the period number, β_1 measures the long-run income elasticity of the money demand and β_2 expresses the semi-elasticity of the money demand with respect to difference

[1] The reasons for introduction of real money gap instead of price gap have been explained in Svensson (2000). We share this view.

between alternative (R^L) and the own rate of interest (R^S)[2]. This difference may be interpreted as an opportunity cost of money holding. Moreover v_0 is a constant term and we expect that $\beta_1 > 0$ and $\beta_2 < 0$.

According to belief that the money demand means demand for the real money and in the economy there is no money illusion, by the specification of the money demand equation (6) for the Polish economy we here assumed that the price elasticity is a unity. This condition will be tested in the further steps.

In steady state, where $\tilde{m}_t = \tilde{m}_t^D$, $y_t = y_t^*$, $v_t = v_t^*$, equation (6) has a form

$$\tilde{m}_t^* = v_0 + \beta_2 y_t^* + \beta_4(R_t^{L*} - R_t^{S*}), \tag{7}$$

where variables R_t^{L*} and R_t^{S*} stand for the long- and short-term equilibrium nominal interest rates (Brzoza-Brzezina, 2003) respectively. Thus the real money gap, which according to the *P-star* concept determines the future inflation may be expressed as

$$\tilde{m}_t - \tilde{m}_t^* = -(p_t - p_t^*) = m_t - p_t - v_0 - \beta_2 y_t^* - \beta_4(R_t^{L*} - R_t^{S*}). \tag{8}$$

Gerlach and Svensson (2003) defined the equilibrium nominal interest rates R_t^{L*} and R_t^{S*} as a sum of the central bank inflation target $\hat{\pi}_t$ and the long- and short-term equilibrium real interest rates respectively. Moreover they assumed that the long- and short-term equilibrium real interest rates are constant. In our opinion in case of the Polish economy in transition this assumption cannot be acceptable.

However the equilibrium real interest rate (natural interest rate) is an unobservable variable and the estimation of such variable is not easy particularly in the transition period. Taking into account all the difficulties and ambivalent results, which have been achieved so far for Poland in transition (Brzoza-Brzezina, 2003) we have simplified the calculations and we assumed that the spread between the actual real long- and short-term interest rates is equal to the spread between the equilibrium values of both rates

$$r_t^{L*} - r_t^{S*} = r_t^L - r_t^S, \tag{9}$$

[2] Own rate of interest for assets included in money is often approximated by the short-term interest rate. Then in the further part of this article "own rate of interest" and "short-term interest rate" will be replaced one by each other. The same concerns "alternative rate of interest" and "long-term interest rate".

where r_t^L i r_t^S stand for the long- and short-term actual real interest rates and r_t^{L*} and r_t^{S*} denotes the equilibrium values of these rates.

Similar to Gerlach and Svensson (2003) we assumed that the equilibrium nominal interest rates are the sum of central bank inflation target and the equilibrium real interest rates. Then the spread between the equilibrium nominal rates has the form

$$R_t^{L*} - R_t^{S*} = (r_t^{L*} + \hat{\pi}_t) - (r_t^{S*} + \hat{\pi}_t) = r_t^{L*} - r_t^{S*}. \tag{10}$$

Next, taking into account (9) and the adaptive nature of the inflation expectations in Poland and assuming that the real interest rate is a sum of the nominal rate and the inflation expectations we may write (10) as

$$R_t^{L*} - R_t^{S*} = r_t^{L*} - r_t^{S*} = r_t^L - r_t^S = (R_t^L - \pi_{t,t-1}^e) - (R_t^S - \pi_{t,t-1}^e) = R_t^L - R_t^S. \tag{11}$$

Equation (11) means that if we accept the assumption that the spread between the actual real long- and short-term interest rates is equal to the spread between the equilibrium values of both rates then the difference between the actual nominal long- and short-term interest rates will be equal to the spread between the equilibrium values of the real rates. Thus the real money gap, which according to the *P-star* concept decides about the future inflation may be written as follows

$$\tilde{m}_t - \tilde{m}_t^* = m_t - p_t - v_0 - \beta_2 y_t^* - \beta_4 (R_t^L - R_t^S). \tag{12}$$

2.3. Inflation model

If the variables $\tilde{m}_t$ and $\tilde{m}_t^*$ are cointegrated (the real money gap forms a cointegrating relation) then under the assumption that the inflation expectations are adaptive, the relation between the future inflation and the real money gap may be expressed in the form (Gerlach and Svensson, 2003)

$$\pi_t = \sum_{q=1}^{Q} \alpha_q \pi_{t-q} + \alpha_m (\tilde{m}_{t-1} - \tilde{m}_{t-1}^*) + \sum_{k=1}^{K} \sum_{i=0}^{I} \theta_{ki} z_{k,t-i} + \varepsilon_t , \tag{13}$$

where $\pi_t = p_t - p_{t-4}$ stands for the inflation rate (the research has been based on the quarterly data and t is a number of the consecutive quarter). Variables z_k denote supply shocks (i.e. oil prices), exchange rate fluctuations and other factors having a short-term impact on the inflation. If the concept of the real money gap is true we may expect that α_m should be within the range from 0 to 2 ($0 < \alpha_m < 2$)[3].

Equation (13) allows concluding that the major factor in the *P-star* concept, which implicates that in case of absence of the supply shocks the inflation expectations differ from the actual inflation is the disequilibrium between the actual real money and its long-term equilibrium value.

However two conditions have to be fulfilled to use the real money gap as an indicator of the future inflation. First the stable long-run money demand function must exist. Secondly the actual real money and its theoretical value must form a long-run equilibrium relationship. The presence of such relationship implicates that the deviation of the real money from its equilibrium level will be temporary and despite the short-term fluctuations the system will return to the steady state.

If the hypothesis about the stable relationship between $\tilde{m}_t$ and $\tilde{m}_t^*$ is true both variables must be cointegrated what means that the error term η_t in the equation

$$\tilde{m}_t = \tilde{m}_t^* + \eta_t , \qquad (14)$$

must be stationary.

Next part of the paper will present the seasonal cointegration analysis of the actual real money and its theoretical equilibrium value $\tilde{m}_t^*$ derived upon the long-run money demand function in the Polish economy in years 1994-2003.[4]

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[3] The expected range for parameter α_m results from the fact that the dependent variable in (13) is the first difference of p, but equation (8) has been normalized in respect to m not p. However the unit price elasticity of money demand, which has been assumed in equation (8), means that parameter at variable p in (8) is equal to –1. Then if the system is stable the adjustment coefficient α_m in (13) should have the opposite sign then in case of error correction mechanism where long-run equation has been normalized in respect to the variable, which stands for dependent variable in short-term equation (Charemza and Deadman, 1997).

[4] Due to the small sample the number of variables in the system must be limited. For that reason while estimating the parameters of the equation (13) we have omitted the exogenous variables, which have only a short-term impact on the inflation. The omitting of the variables having a short-term impact on the inflation is a strong assumption because in the sample, which covers only 10 years these variable may play an important role in the description of the inflation behaviour. Thus we want to emphasise that the purpose of this research was not to formulate a model with the perfect short-term forecast ability but to investigate whether the real money gap may be the leading indicator for the future inflation.

3. Seasonal cointegration methodology

3.1. The concept of seasonal cointegration

The seasonal cointegration is a generalisation of the classic concept of cointegration proposed in the seminal work of Engle and Granger (1987) [5].

In case when two or more variables are nonstationary because of the presence of the nonseasonal and seasonal unit roots (Hylleberg *et al.*, 1990) we may consider cointegrating relations for these variables not only at zero frequency but also at other two frequencies connected with seasonal fluctuations. The cointegrating relation at zero frequency may be interpreted as the long-run equilibrium relation while the cointegration at seasonal frequencies may be explained as the "parallel" common movements in the seasonal components of the variables (Engle *et al.*, 1993).[6]

Formally, the definition of the seasonal cointegration may be written as follows (Engle *et al.*, 1993):

The variables $y_{1t}, y_{2t},..., y_{Kt}$, composed in the vector $\mathbf{y}_t$ are cointegrated of order (d, b) at the seasonal frequency θ_S, where $d \geq b > 0$, briefly $\mathbf{y}_t \sim CI_{\theta_S}(d, b)$, if all components of $\mathbf{y}_t$ are $I_{\theta_S}(d)$ and there exists a non-zero vector $\mathbf{c} = (c_1, c_2,...,c_K)'$, that $\mathbf{c}'\mathbf{y}_t \sim I_{\theta_S}(d-b)$. This vector $\mathbf{c}$ is called the cointegrating vector at frequency θ_S.[7]

For the analyse of the system of economic variables the most interesting is the case when $d = b$, which means that the deviations from the steady state are stationary, because the coefficients composed in cointegrating vectors at seasonal frequencies may be interpreted as the parameters of medium-term economic relations between variables of interest. This case will be considered further in this work.

[5] In the article we use a term of the stochastic process $\{Y_t\}$, defined as a set of stochastic variables ordered according to the time index $t = 1,2...,T$. Moreover we introduce the term of the time series $\{y_t\}$, defined as a realisation of the stochastic process $\{Y_t\}$ in a particular sample. To simplify the notation these terms will be sometimes replaced one by each other and denoted by one symbol y_t. According to these rules symbol $\mathbf{y}_t$ stands for both multivariate stochastic process (multivariate time series) and vector of the stochastic variables. The elements of this vectors will be denoted as $y_{k,t}$, where $k = 1, 2,...,K$.

[6] According to the definition proposed primary by Hylleberg *et al.* (1990) seasonal cointegration means the cointegration only at seasonal frequencies. However in many theoretical and practical works (Johansen and Schaumburg, 1999), the term "seasonal cointegration analysis" corresponds to the cointegration analysis conducted at not only seasonal but nonseasonal frequencies as well. We will use this term in this more general meaning.

[7] It is noteworthy that for some frequencies the elements of cointegrating vector may be the complex numbers.

3.2. Seasonal error correction model

The appropriate tool for the seasonal cointegration analysis is the multivariate approach proposed primary by Lee (1992) and developed by Franses and Kunst (1999) and Johansen and Schaumburg (1999). This algorithm is a generalisation of the Johansen method (Johansen, 1995) widely used in the cointegration analysis at zero frequency.

In the method proposed by Lee (1992) the starting point is a vector autoregressive (VAR) model without constraints imposed on the parameters. This model may be written as

$$\mathbf{y}_t = \mathbf{v} + \mathbf{A}_1\mathbf{y}_{t-1} + \mathbf{A}_2\mathbf{y}_{t-2} + \ldots + \mathbf{A}_p\mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t, \text{ dla } t = 1,2,\ldots,T \tag{15}$$

or in a lag operator notation as

$$\mathbf{A}(L)\mathbf{y}_t = \boldsymbol{\varepsilon}_t, \tag{16}$$

where $\mathbf{A}_i$ are fixed $K \times K$ coefficient matrices, $\mathbf{y}_t = [y_{1t}, y_{2t}, \ldots, y_{Kt}]'$ is a vector of K endogenous variables in (15), $\mathbf{v}_t = [v_{1t}, v_{2t}, \ldots, v_{Kt}]'$ is a vector of the deterministic terms (intercept, deterministic trend, seasonal dummies), $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}, \varepsilon_{2t}, \ldots, \varepsilon_{Kt}]'$ is a K-dimensional white noise process while p stands for a lag length in model (15). In (16) $\mathbf{A}(L) = \mathbf{I} - \mathbf{A}_1 L - \mathbf{A}_2 L^2 - \ldots - \mathbf{A}_p L^P$ denotes a $K \times K$ matrix lag polynomial. Moreover we assume that for $t \leq 0$ the initial values $\mathbf{y}_t$ are fixed.

According to Lagrange expansion of the polynomial $\mathbf{A}(z)$ around the points $z_1, z_2, \ldots, z_S$ VAR model (16) may be written as follows (Johansen and Schaumburg, 1999)

$$\mathbf{A}(L)\mathbf{y}_t = (p(L)\mathbf{I} + \sum_{k=1}^{S} \mathbf{A}(z_k)\frac{p_k(L)L}{p_k(z_k)z_k} + p(L)L\mathbf{A}_0(L))\mathbf{y}_t = \boldsymbol{\varepsilon}_t, \tag{17}$$

what gives

$$p(L)\mathbf{y}_t = -\sum_{k=1}^{S} \mathbf{A}(z_k)\frac{p_k(L)L}{p_k(z_k)z_k}\mathbf{y}_t - p(L)\mathbf{A}_0(L)L\mathbf{y}_t + \boldsymbol{\varepsilon}_t, \tag{18}$$

where $\mathbf{A}(L)$, $\mathbf{A}_0(L)$, $p(L)$ i $p_k(L)$ are the lag polynomials, from which

$$p(L) = \prod_{k=1}^{S}(1-\frac{1}{z_k}L) \quad \text{and} \quad p_k(L) = p(L)/(1-\frac{1}{z_k}L).$$

If $z_1, z_2, \ldots, z_S$ are the roots of equation $\det\mathbf{A}(z) = 0$, then the matrices $\mathbf{A}(z_k)$ are singular (of reduced rank).

According to Granger Representation Theorem (Johansen and Schaumburg, 1999), if the matrix $\mathbf{A}(z_k)$ has a rank r_k, where $0 < r_k < K$, this matrix may be written as a product of two matrices (real or complex) of dimension $p \times r_k$ and rank r_k

$$\mathbf{A}(z_k) = -\boldsymbol{\alpha}_k \boldsymbol{\beta}_k^{'}, \tag{19}$$

where $\boldsymbol{\alpha}_k$ i $\boldsymbol{\beta}_k$ are the nonzero matrices of rank r_k. After substitution for $\mathbf{A}(z_k)$ from (19) equation (18) may be expressed as

$$p(L)\mathbf{y}_t = \sum_{k=1}^{S}\boldsymbol{\alpha}_k\boldsymbol{\beta}_k' \frac{p_k(L)L}{p_k(z_k)z_k}\mathbf{y}_t - p(L)\mathbf{A}_0(L)L\mathbf{y}_t + \boldsymbol{\varepsilon}_t =$$

$$= \sum_{k=1}^{S}\boldsymbol{\alpha}_k\boldsymbol{\beta}_k'\mathbf{y}_t^{(k)} - p(L)\mathbf{A}_0(L)\mathbf{y}_{t-1} + \boldsymbol{\varepsilon}_t, \tag{20}$$

where $\mathbf{y}_t^{(k)} = \frac{p_k(L)L}{p_k(z_k)z_k}\mathbf{y}_t$.

The model (20) is called the seasonal error correction model (SECM.) and it is interpreted as the generalisation of the vector error correction model (VECM), used in the classic (only at zero frequency) cointegration analysis.

For every frequency θ_k the error correction mechanism expressed in terms of the cointegrating matrix $\boldsymbol{\beta}_k$ and the adjustment matrix $\boldsymbol{\alpha}_k$ may exist. The variables included in vector $\mathbf{y}_t^{(k)}$ have been defined in the way that the corresponding time series have only the unit root at frequency θ_k, while the other unit roots have been filtered out.

On the left hand side of (20) all variables composed in $p(L)\mathbf{y}_t$ are stationary, while on the right hand side the variables included in vector $\mathbf{y}_t^{(k)}$ are nonstationary only at frequency θ_k. It means that at frequency θ_k their linear combination $\boldsymbol{\beta}_k^{'}\mathbf{y}_t^{(k)}$ must be stationary which stays in line with a definition of cointegration at frequency θ_k between variables composed in vector $\mathbf{y}_t^{(k)}$. The number of cointegrating relations is equal to the rank of $\mathbf{A}(z_k)$. The cointegrating vectors at frequency θ_k correspond to the subsequent columns of the cointegrating matrix $\boldsymbol{\beta}_k$, and the elements of adjustment matrix $\boldsymbol{\alpha}_k$ measure the speed of adjustment to the equilibrium relation at this frequency.

3.3. Seasonal cointegration for quarterly data

The seasonal error correction model for quarterly data may be derived by expanding the polynomial $\mathbf{A}(L)$ given in (16) around the roots 1, –1, –i and i. The root equal to 1 corresponds to zero (long-run) frequency ($\theta = 0$), the root –1 to biannual frequency ($\theta = \pi$), and the roots i and –i to annual frequency ($\theta = \pi/2$). The cointegration at frequency $\theta = \pi$ may be interpreted as the common changes of a biannual seasonality pattern, and at frequency $\theta = \pi/2$ as the common movements of the quarterly seasonality pattern. For quarterly data (20) has a form

$$\Delta_4 \mathbf{y}_t = \mathbf{y}_t - \mathbf{y}_{t-4} = \sum_{k=1}^{4} \boldsymbol{\alpha}_k \boldsymbol{\beta}'_k \mathbf{y}_t^{(k)} + \sum_{i=1}^{p-4} \boldsymbol{\Gamma}_i \Delta_4 \mathbf{y}_{t-i} + \boldsymbol{\varepsilon}_t, \tag{21}$$

where

$$\mathbf{y}_t^{(1)} = 1/4(\mathbf{y}_{t-1} + \mathbf{y}_{t-2} + \mathbf{y}_{t-3} + \mathbf{y}_{t-4}),$$
$$\mathbf{y}_t^{(2)} = -1/4(\mathbf{y}_{t-1} - \mathbf{y}_{t-2} + \mathbf{y}_{t-3} - \mathbf{y}_{t-4}),$$
$$\mathbf{y}_t^{(3)} = (1/4i)(\mathbf{y}_{t-1} - i\mathbf{y}_{t-2} - \mathbf{y}_{t-3} + i\mathbf{y}_{t-4}) = -(1/4)(\mathbf{y}_{t-2} - \mathbf{y}_{t-4}) - (i/4)(\mathbf{y}_{t-1} - \mathbf{y}_{t-3}),$$
$$\mathbf{y}_t^{(4)} = -(1/4i)(\mathbf{y}_{t-1} + i\mathbf{y}_{t-2} - \mathbf{y}_{t-3} - i\mathbf{y}_{t-4}) = -(1/4)(\mathbf{y}_{t-2} - \mathbf{y}_{t-4}) + (i/4)(\mathbf{y}_{t-1} - \mathbf{y}_{t-3}).$$

Despite the fact that the components of $\mathbf{y}_t^{(3)}$ and $\mathbf{y}_t^{(4)}$ are the complex numbers the seasonal error correction model given by (21) may be written using only the real numbers. Then we express the variables $\mathbf{y}_t^{(3)}$ and $\mathbf{y}_t^{(4)}$ as $\mathbf{y}_t^{(3)} = \mathbf{y}_{Rt} + i\mathbf{y}_{It}$ and $\mathbf{y}_t^{(4)} = \bar{\mathbf{y}}^{(3)} = \mathbf{y}_{Rt} - i\mathbf{y}_{It}$, where

$$\mathbf{y}_{Rt} = -1/4(\mathbf{y}_{t-2} - \mathbf{y}_{t-4}),$$
$$\mathbf{y}_{It} = -1/4(\mathbf{y}_{t-1} - \mathbf{y}_{t-3}).$$

On the left hand side of (21) there are only the real numbers and the variables in vectors $\mathbf{y}_t^{(3)}$ and $\mathbf{y}_t^{(4)}$ come in complex conjugate pairs. This means that the matrices $\boldsymbol{\alpha}_3$, $\boldsymbol{\alpha}_4$ and $\boldsymbol{\beta}_3$, $\boldsymbol{\beta}_4$ must be in complex conjugate pairs as well. The relationships between the adjustment and the cointegrating matrices may be written as follows

$$\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_R + i\boldsymbol{\alpha}_I = \bar{\boldsymbol{\alpha}}_4 \qquad \text{and} \qquad \boldsymbol{\beta}_3 = \boldsymbol{\beta}_R - i\boldsymbol{\beta}_I = \bar{\boldsymbol{\beta}}_4 .$$

After substitution for $\boldsymbol{\alpha}_3$ and $\boldsymbol{\alpha}_4$ as well for $\boldsymbol{\beta}_3$ and $\boldsymbol{\beta}_4$ the error correction mechanism at annual frequency may be expressed in the form

$$\begin{aligned}
&\boldsymbol{\alpha}_3\boldsymbol{\beta}'_3\mathbf{y}_t^{(3)} + \boldsymbol{\alpha}_4\boldsymbol{\beta}'_4\mathbf{y}_t^{(4)} = \\
&= (\boldsymbol{\alpha}_R + i\boldsymbol{\alpha}_I)(\boldsymbol{\beta}'_R - i\boldsymbol{\beta}'_I)(\mathbf{y}_{Rt} + i\mathbf{y}_{It}) + (\boldsymbol{\alpha}_R - i\boldsymbol{\alpha}_I)(\boldsymbol{\beta}'_R + i\boldsymbol{\beta}'_I)(\mathbf{y}_{Rt} - i\mathbf{y}_{It}) = \\
&= \boldsymbol{\alpha}_R\boldsymbol{\beta}'_R\mathbf{y}_{Rt} - i\boldsymbol{\alpha}_R\boldsymbol{\beta}'_I\mathbf{y}_{Rt} + i\boldsymbol{\alpha}_I\boldsymbol{\beta}'_R\mathbf{y}_{Rt} + \boldsymbol{\alpha}_I\boldsymbol{\beta}'_I\mathbf{y}_{Rt} + i\boldsymbol{\alpha}_R\boldsymbol{\beta}'_R\mathbf{y}_{It} + \boldsymbol{\alpha}_R\boldsymbol{\beta}'_I\mathbf{y}_{It} - \boldsymbol{\alpha}_I\boldsymbol{\beta}'_R\mathbf{y}_{It} + i\boldsymbol{\alpha}_I\boldsymbol{\beta}'_I\mathbf{y}_{It} + \\
&+ \boldsymbol{\alpha}_R\boldsymbol{\beta}'_R\mathbf{y}_{Rt} + i\boldsymbol{\alpha}_R\boldsymbol{\beta}'_I\mathbf{y}_{Rt} - i\boldsymbol{\alpha}_I\boldsymbol{\beta}'_R\mathbf{y}_{Rt} + \boldsymbol{\alpha}_I\boldsymbol{\beta}'_I\mathbf{y}_{Rt} - i\boldsymbol{\alpha}_R\boldsymbol{\beta}'_R\mathbf{y}_{It} + \boldsymbol{\alpha}_R\boldsymbol{\beta}'_I\mathbf{y}_{It} - \boldsymbol{\alpha}_I\boldsymbol{\beta}'_R\mathbf{y}_{It} - i\boldsymbol{\alpha}_I\boldsymbol{\beta}'_I\mathbf{y}_{It},
\end{aligned}$$

what gives

$$\begin{aligned}
&\boldsymbol{\alpha}_3\boldsymbol{\beta}'_3\mathbf{y}_t^{(3)} + \boldsymbol{\alpha}_4\boldsymbol{\beta}'_4\mathbf{y}_t^{(4)} = 2\boldsymbol{\alpha}_R\boldsymbol{\beta}'_R\mathbf{y}_{Rt} + 2\boldsymbol{\alpha}_I\boldsymbol{\beta}'_I\mathbf{y}_{Rt} - 2\boldsymbol{\alpha}_R\boldsymbol{\beta}'_I\mathbf{y}_{It} + 2\boldsymbol{\alpha}_I\boldsymbol{\beta}'_R\mathbf{y}_{It} = \\
&= 2(\boldsymbol{\alpha}_R\boldsymbol{\beta}'_R + \boldsymbol{\alpha}_I\boldsymbol{\beta}'_I)\mathbf{y}_{Rt} + 2(\boldsymbol{\alpha}_R\boldsymbol{\beta}'_I - \boldsymbol{\alpha}_I\boldsymbol{\beta}'_R)\mathbf{y}_{It} = \\
&= -1/2(\boldsymbol{\alpha}_R\boldsymbol{\beta}'_R + \boldsymbol{\alpha}_I\boldsymbol{\beta}'_I)(\mathbf{y}_{t-2} - \mathbf{y}_{t-4}) + 1/2(\boldsymbol{\alpha}_I\boldsymbol{\beta}'_R - \boldsymbol{\alpha}_R\boldsymbol{\beta}'_I)(\mathbf{y}_{t-1} - \mathbf{y}_{t-3}).
\end{aligned}$$

The seasonal error correction model (21) may now be written using only real variables (for simplification of notation the factors 1/2 have been included in the appropriate adjustment matrices)

$$\begin{aligned}
&\Delta_4\mathbf{y}_t = \boldsymbol{\alpha}_1\boldsymbol{\beta}'_1\mathbf{y}_t^{(1)} + \boldsymbol{\alpha}_2\boldsymbol{\beta}'_2\mathbf{y}_t^{(2)} - (\boldsymbol{\alpha}_R\boldsymbol{\beta}'_R + \boldsymbol{\alpha}_I\boldsymbol{\beta}'_I)(\mathbf{y}_{t-2} - \mathbf{y}_{t-4}) + \\
&+ (\boldsymbol{\alpha}_I\boldsymbol{\beta}'_R - \boldsymbol{\alpha}_R\boldsymbol{\beta}'_I)(\mathbf{y}_{t-1} - \mathbf{y}_{t-3}) + \sum_{i=1}^{p-4}\boldsymbol{\Gamma}_i\Delta_4\mathbf{y}_{t-i} + \boldsymbol{\varepsilon}_t.
\end{aligned} \qquad (22)$$

In (22) the coefficient matrices at $(\mathbf{y}_{t-1} - \mathbf{y}_{t-3})$ and its lag $(\mathbf{y}_{t-2} - \mathbf{y}_{t-4})$ are rather complicated. Moreover they even don't need to have the reduced rank. That is why Ghysels and Osborn (2001) suggest expressing the equation (22) in a slightly modified form

$$\Delta_4\mathbf{y}_t = \boldsymbol{\alpha}_1\boldsymbol{\beta}'_1\mathbf{y}_t^{(1)} + \boldsymbol{\alpha}_2\boldsymbol{\beta}'_2\mathbf{y}_t^{(2)} - \boldsymbol{\alpha}_R[\boldsymbol{\beta}'_R(\mathbf{y}_{t-2} - \mathbf{y}_{t-4}) + \boldsymbol{\beta}'_I(\mathbf{y}_{t-1} - \mathbf{y}_{t-3})] +$$

$$+ \boldsymbol{\alpha}_I[\boldsymbol{\beta}'_R(\mathbf{y}_{t-1} - \mathbf{y}_{t-3}) - \boldsymbol{\beta}'_I(\mathbf{y}_{t-2} - \mathbf{y}_{t-4})] + \sum_{i=1}^{p-4} \boldsymbol{\Gamma}_i \Delta_4 \mathbf{y}_{t-i} + \boldsymbol{\varepsilon}_t. \quad (23)$$

This way the interpretation of the adjustment mechanism is more straightforward. The equation (23) implies that both linear combinations $\boldsymbol{\beta}'_R(1 - L^2)\mathbf{y}_t$ and $\boldsymbol{\beta}'_I(1 - L^2)\mathbf{y}_t$ are stationary or they are nonstationary but cointegrated with each other.

The first case allows concluding that the components of vector $(1 - L^2)\mathbf{y}_t$ are cointegrated at annual frequency and the columns of $\boldsymbol{\beta}_R$ and $\boldsymbol{\beta}_I$ are the cointegrating vectors. In the second case the elements of vector $(1 - L^2)\mathbf{y}_t$ are cointegrated with their own lags what corresponds to the so called *polynomial cointegration*.

Regarding the fact that the interpretation of the error correction mechanism at annual frequency $\pi/2$ in (23) is still complicated Johansen and Schaumburg (1999) proposed to simplify the mechanism by bounding the cointegrating matrices only to real numbers. This assumption ($\boldsymbol{\beta}_I = \mathbf{0}$) allows expressing the part of model (23) connected with annual frequency in the form

$$\boldsymbol{\alpha}_I\boldsymbol{\beta}'_R(\mathbf{y}_{t-1} - \mathbf{y}_{t-3}) - \boldsymbol{\alpha}_R\boldsymbol{\beta}'_R(\mathbf{y}_{t-2} - \mathbf{y}_{t-4}).$$

Then, if at annual frequency the variables in vector $\mathbf{y}_t$ are integrated of order 1 and the above mentioned expression is stationary, two scenarios are possible. The linear combination $\boldsymbol{\beta}'_R(1 - L^2)\mathbf{y}_t$ is stationary and the cointegrating vectors are columns of $\boldsymbol{\beta}_R$, or this combination is cointegrated with its own lag $\boldsymbol{\beta}'_R(1 - L^2)L\mathbf{y}_t$. This second variant means that between variables in $\mathbf{y}_t$ exists the polynomial cointegration. The assumption about the real cointegrating matrix ($\boldsymbol{\beta}_I = \mathbf{0}$) will be verified further in this article (section 3.5.6).

3.4. Testing for the rank of cointegration

We used the algorithm proposed firstly be Lee (1992) and developed by Franses and Kunst (1999) and Johansen and Schaumburg (1999) for the parameter estimation of the seasonal error correction model (23) – SECM. This algorithm is the Full Information Maximum Likelihood (FIML) method and its detailed description may be found in the paper by Johansen and Schaumburg (1999). While estimating the parameters of SECM we need to know the rank of cointegration (number of cointegrating vectors) at all frequencies. In order to find the rank of cointegration Lee (1992) proposed a likelihood ratio test known as *Johansen trace test*.

Taking into account that the variables in vectors $\mathbf{y}_t^{(k)}$ are asymptotically uncorrelated (Johansen and Schaumburg, 1999) in the sense that

$$T^{-2}\sum_{t=1}^{T}\mathbf{y}_t^{(j)}\mathbf{y}_t^{(k)/} \xrightarrow{P} 0 \text{ for } j \neq k, \tag{24}$$

the cointegration rank may be tested at every frequency θ_k ignoring the number of cointegrating vectors at other frequencies.[8]

In the likelihood ratio test proposed by Lee (1992) the maximum of the likelihood function for the model with cointegration rank r_0 at frequency θ_k is compared with the maximum of the likelihood function for the model where matrix $\mathbf{A}(z_k)$ defined by (19) is of the full rank K (which means that the variables in $\mathbf{y}_t^{(k)}$ are stationary at frequency θ_k).

The sequence of the hypotheses is the same at all frequencies and the same as in the "classic" concept of the cointegration (Johansen, 1995). The null hypothesis assumes that the number of cointegrating vectors at frequency θ_k denoted as r_k equals to r_0

H_0: $r_k = r_0$,

against the alternative hypothesis that the rank of the matrix $\mathbf{A}(z_k)$ is equal to K (in case of annual frequency the rank of both matrices corresponding to this frequency amounts to K).

Thus at every frequency the asymptotic distribution of the test statistic is nonstandard and the critical values (depending on the difference $K - r_k$) have been derived in Monte Carlo simulations. The appropriate tables with critical values may be found in Johansen and Schaumburg (1999) and Franses and Kunst (1999).

The testing scheme is the same for all frequencies. We start from the null hypothesis H_0: $r_k = 0$ and if the value of test statistic is larger than the critical value we must reject H_0 in favour of H_1 and after that test the succeeding hypotheses

H_0: $r_k = 1$ against H_1: $r_k > 1$

H_0: $r_k = 2$ against H_1: $r_k > 2$

[8] It is noteworthy that the proposed method of estimation has been based on the assumption that variables $\mathbf{y}_t^{(k)}$, for $k = 1,2,3,4$ are uncorrelated. However this assumption holds only asymptotically what in the sample of 40 observations may lead to a biasness of estimators. Thus Cubadda and Omtzigt (2003) show that even in the small samples this biasness doesn't have to be large.

...

H_0: $r_k = K - 1$ against H_1: $r_k = K$.

The testing sequence will be terminated when the test statistic will be lower than the critical value. If the hypothesis H_0: $r_k = r_0 - 1$ is rejected and the next hypothesis H_0: $r_k = r_0$, cannot be rejected the number of the cointegrating vectors will be chosen to r_0.

4. Empirical results for Poland

4.1. Data description

The *P-star* model presented in part 2 of this work has been verified for Polish data from the period 1994 – 2003 using the estimation method proposed in part 3.

As a variable representing the price level in the *P-star* model we proposed the consumer price index[9] (variable *CPI*) and for the money variable we employed the broad monetary aggregate M3 (variable *M*3). The choice of the appropriate money variable has been based on the results of research for other European countries (Gerlach and Svensson, 2003), where the best results have been achieved for M3 aggregate.

The long-term interest rate is a yield of 5-years T-bonds (variable *R_L*) while 3-months WIBOR rate (Warsaw Interbank Rate – denoted as *R_S*) is considered as the short-term rate. Certainly the average yield of assets included in money would be a better measure but regarding the lack of proper data we must employ the interbank rate (WIBOR). According to (6) the short- and long-term interest rates have been expressed as the difference denoted in this paper as *SPREAD*.

Potential output in constant prices (*POT*) has been derived upon the so-called LRRO method (*long-run restrictions applied to output*) and the detailed results may be found in the paper by Kotłowski (2003).

All data used in the research are quarterly data from 1994 Q1 to 2004 Q4 (40 observations). The GDP time series in constant prices from 1995[10] and the index of consumer prices has been taken from GUS (Central Statistical Office) database. M3 time series come from NBP (National Bank of Poland) releases. 3-months WIBOR is published by Reuters and the average yield of T-bonds has been calculated on the basis of Reuters and Ministry of

[9] Expressed as an index on the basis of value at constant prices (quarterly average of 1994 = 100).
[10] The estimates of quarterly GDP for 1994 have been taken from IBnGR release.

Finance data. The entire variables with the exception of interest rates have been expressed in logs.

4.2. Testing of order of integration

In the first stage of the research we have carried out the HEGY test (Hylleberg *et al.*, 1990) to investigate the seasonal structure of the variable that we use in the *P-star* model. In the first step we must fix the maximum order of integration at every frequency. Considering the results of the analogical works for other European countries (Juselius, 2002) we assumed that with the exception of *cpi* all other variables have no more than one unit root at every frequency.

In case of *cpi* we assumed that the maximum order of integration at zero frequency may be equal to 2. Juselius (2002) emphasised that the maximum order of integration for the variables representing the price level may depend on the country and the length of the sample and should be tested.

For that reason for the *cpi* variable we have chosen the testing strategy, which encompasses the case that maximum order of integration may be higher then one. This strategy has been described in details by Ghysels and Osborn (2001) and assumes that HEGY test must be applied not for the levels of the variable but for its first differences. It means that, by assumption, the *cpi* time series have one unit root at zero frequency and in the HEGY test we investigate the presence of the second such unit root.

If the test statistic will be larger than the appropriate critical value and the null hypotheses (about existence of the second root) will not be rejected the conclusion is such that at zero frequency this time series have two unit roots equal to 1.

In the opposite case, if the null hypothesis will be rejected then using the ADF test we will prove the hypothesis that at zero frequency the *cpi* variable is integrated of order one against the alternative hypothesis that the *cpi* variable is stationary at this frequency. In the auxiliary regression in the ADF test we will not use the first differences of the *cpi* as in the standard version of this test (Charemza and Deadman, 1997) but the variable filtered out from not only unit root equal to 1 but also from the other roots identified in the first sequence of the testing procedure (HEGY test for the first differences – see table 2).

In case of the other variables the HEGY test has been applied directly to the levels of the variables, which means that the highest order of integration may be equal to one.

The results of the HEGY test for all variables have been presented in the table 1 while the table 2 presents the results of the additional ADF test for the presence of the unit root for the variable *cpi*.

Table 1. The results of tests for the presence of the seasonal and nonseasonal unit roots

Variable	Deterministic components	t_{π_1}	Critical values (5%)	t_{π_2}	Critical values (5%)	$F_{\pi_3 \cap \pi_4}$	Critical values (5%)
pot	intercept	–2.295	–2.86 –2.78	0.142	–1.87 –1.79	0.049	3.61 3.82
Δ*cpi*	intercept	–2.968**	–2.86 –2.78	–2.842**	–1.87 –1.79	2.419	3.61 3.82
*m*3	seasonal dummies	–2.184	–2.78 –2.69	–3.173**	–2.78 –2.69	31.05**	9.46 10.1
	seasonal dummies	–0.159	–2.78 –2.69	–2.844**	–2.78 –2.69	11.99**	9.46 10.1

(**) denotes rejection of the null hypothesis at 5% significance level. For every critical value we presented two numbers – lower and upper limit. If the test statistic in the test for π_1 and π_2 is smaller than lower critical value (larger then upper critical value in the test for π_3 and π_4) the null hypothesis about the unit root must be rejected. If the test statistic in the test for π_1 and π_2 is larger than upper critical value (smaller than lower critical value in the test for π_3 and π_4) the null hypothesis about the unit root cannot be rejected. Within the range between the lower and upper critical values any decision cannot be taken. The critical values in HEGY test have been obtained from Charemza and Deadman (1997).

Table 2. The results of test for the presence of unit root at zero frequency (variable *cpi*)

H_0	H_1	t_{π_1}	Critical value (5%)
$(1–L+L^2–$	$(1+L^2)CPI \sim I(0)$	–2.778	–2.968

(**) denotes rejection of the null hypothesis at 5% significance level. The critical values in the HEGY test come from MacKinnon (1996).

1) For all variables (with the exception of Δ*cpi*) the results of the HEGY test indicate that the null hypothesis about the presence of a unit root equal to 1 cannot be rejected. In case of Δ*cpi* the test statistic t_{π_1} is smaller than the critical value, which results in rejection of the null hypothesis about two unit roots at zero frequency. Moreover the results of the additional ADF test (Table 2) allow concluding that at 5% significance level the null hypothesis about one unit root 1 cannot be rejected. It means that the variable *cpi* like other variables in the *P-star* model is integrated of order one at zero frequency. The results indicate that all four variables show the stochastic trend and we may search for the long-run equilibrium relation between these variables (cointegration vector at zero frequency).
2) The presence of the unit root equal to –1 corresponding to biannual fluctuations has been confirmed only for the potential output (variable *pot*). The results of this test sequence imply that in case of the *cpi*, *m*3 and *SPREAD* variables the biannual seasonality doesn't exist or if it exists it is the deterministic or stationary stochastic seasonality.

3) For variables *m*3 and *SPREAD* we must reject the hypothesis about the complex roots equal to *i* and –*i*.

It is noteworthy that for *m*3, which represents money holdings the test results didn't confirm any presence of the seasonal unit roots at 5% significance level for both frequencies. Thus at 1% level the results indicate the presence of the seasonal unit root at biannual frequency. Moreover at this significance level a unit root at biannual frequency has been confirmed for all variables in the *P-star* model.

Taking into account the presence of the seasonal unit root for the money variable at 1% significance level and the presence of the seasonal unit roots at 5% level for all other variables we think that the application of the seasonal cointegration method to prove the adequacy of the *P-star* model in the Polish circumstances may be useful. Franses and Löf (2000) show the advantages of the seasonal error correction models as compared with the "classic" (nonseasonal) error correction models.

4.3. Seasonal cointegration analysis

4.3.1. The specification of the VAR model

The basis for the seasonal error correction model (SECM) is a vector autoregressive (VAR) model defined in (15) with four endogenous variables (money, prices, potential output and interest rate spread), intercept and three dummy variables, which correspond to first three quarters[11].

The order of the VAR model has been determined on the basis of the AIC and SC information criteria. Moreover we have tested the autocorrelation of the error term. Taking into account the short sample we have assumed that the order of the VAR model cannot be larger than 5. For that reason the choice was made between order equal to 4 (no lags for endogenous variables in SECM) and order equal to 5 (one lag in SECM).

In our case both criteria indicated the VAR order equal to 5. This choice has been supported by the autocorrelation LM test, where the test statistic confirmed no autocorrelation of order 1 and 4 at 10% significance level.

[11] Among the possible variants of the seasonal error correction models, which differ in the deterministic components (Johansen and Schaumburg, 1999) we have chosen the model where the seasonal intercepts have been restricted to the cointegrating space while a zero frequency intercept is unrestricted. The choice of this variant of the SECM was made due to two reasons. First our intention was to assure the presence of the additional deterministic seasonality in the model and secondly all variables in the model exhibit a trend.

4.3.2. The determination of the cointegration rank on the basis of the companion matrix

After the specification of the VAR model we have calculated the so called companion matrix and we have preliminary determined the number of the cointegrating vectors in the model (Juselius, 2002). The analysis of the companion matrix corresponds to the stability assessment of the whole system (VAR model). The eigenvalues of the companion matrix are the inverses of the roots of the characteristic polynomial for (16) and if the system is stable all eigenvalues should be inside the unit circle. The number of eigenvalues equal in modulus to 1 determines the number of so called common trends in the system. By subtracting the number of common trends from the number of endogenous variables we obtain the number of cointegrating vectors (Johansen, 1995, Juselius, 2002).

The real eigenvalues equal to 1 correspond to the common trends at zero frequency, eigenvalues equal to –1 are connected with biannual frequency and the number of complex conjugate pairs *i* and –*i* determine the number of common trends at annual frequency.

Table 3 contains the eigenvalues of the companion matrix for which the modulus value was larger than 0.8. We assumed that these eigenvalues for which the real or imaginary part is larger than 0.85 correspond to the common trends in the system.

Table 3. The eigenvalues of the companion matrix larger than 0.8 in modulus

Frequency	Roots	Modulus
$\theta = 0$	1.026903 ± 0.222003i	1.050626
	0.957832	0.957832
	0.694533 ± 0.601606i	0.918861
	0.783352 ± 0.363965i	0.863777
$\theta = \pi$	–0.657790 ± 0.603693i	0.892823
	–0.864419 ± 0.150200i	0.877371
$\theta = \pi/2$	0.338876 ± 0.828850i	0.895449
	–0.135618 ± 0.876254i	0.886686

In case of the zero frequency the results indicate three common trends what implies only one cointegrating vector. At biannual frequency the number of the common trends allows concluding about two cointegrating vectors. Finally at annual frequency we identified two common trends which in case of this frequency indicates three pairs of complex conjugate cointegrating vectors.[12].

[12] According to (21), the error correction mechanism at annual frequency contains two cointegrating matrices: matrix $\boldsymbol{\beta}_3 = \boldsymbol{\beta}_R - i\boldsymbol{\beta}_I$ connected with a unit root equal to *i* and matrix $\boldsymbol{\beta}_3 = \boldsymbol{\beta}_R + i\boldsymbol{\beta}_I$ corresponding to a root equal to –*i*. It means that the cointegrating vectors at annual frequency come in conjugate pairs and the number of these pairs correspond to rank of cointegration. For that reason, if we want to calculate the cointegration rank at annual frequency (the number of complex conjugate vectors), then from the number of endogenous variables

Concluding about the number of cointegration vectors using the companion matrix we have to remember that all eigenvalues have been calculated on the basis of the parameter estimates not on the true parameters. Moreover, the choice of the break even values to classify the eigenvalues as corresponding to the common trend is a subjective decision. This is why we should treat the obtained results only as a hint and the final decision should be based on the statistical tests. The results of these tests will be presented in the next section.

4.3.3. Testing for the cointegration rank

The inference about the rank of cointegration in the *P-star* model has been conducted upon the trace test described in the section 3.4. Considering the short length of the sample the crucial problem was to choose the critical values. Greenslade *at al*. (2002) emphasise that testing for the cointegration rank on the basis of the asymptotic values leads to an overestimation of the true number of the cointegrating vectors. On the other hand the use of the small sample critical values may result in an underestimation of the true rank of cointegration. For that reason we have used both sets of the critical values. The table 4 contains the results of the trace test with the asymptotic critical values[13] and in the table 5 the results using the small sample critical values[14] have been presented.

Table 4. Testing for the cointegration rank with the asymptotic critical values

H_0	$\theta = 0$			$\theta = \pi$			$\theta = \pi/2$		
	Statistic value	Critical value		Statistic value	Critical value		Statistic value	Critical value	
		10%	5%		10%	5%		10%	5%
r = 0	94.34**	43.84	47.21	53.30**	49.5	53.0	146.75**	95.60	100.45
r = 1	44.62**	26.70	29.38	25.67	30.9	33.6	78.30**	60.00	63.90
r = 2	11.81	13.31	15.34	11.35	17.0	19.2	34.10*	32.48	35.47
r = 3	0.95	2.71	3.84	4.77	7.2	8.7	12.81	13.07	15.12

(*) and (**) denotes rejection of the null hypothesis at 10% and 5% significance level.

we must subtract the number of common trends divided by 2. This is why for 4 endogenous variables and 2 common trends the cointegration rank has been determined as 3.

[13] The asymptotic critical values at zero frequency come from the table 1* in Osterwald-Lenum (1992), at biannual frequency from tables 1a – 1d in Franses and Kunst (1999) while at annual frequency from table 3 in Johansen and Schaumburg (1999). It is noteworthy that the critical values tabulated in Franses and Kunst (1999) are not the asymptotic ones. They have been simulated for the sample of 100 observations. However in our research, where the sample covers only 40 observations these values are near to the asymptotic ones and for the simplicity they will be called asymptotic.

[14] The small sample critical values have been calculated in the Monte Carlo simulations similarly to the experiment proposed in Franses and Kunst (1999). The critical values are based on 20.000 replications using the model (22) with a constant. The sample covered 40 observations.

Table 5. Testing for the cointegration rank with the small sample critical values

H_0	$\theta = 0$			$\theta = \pi$			$\theta = \pi/2$		
	Statistic value	Critical value		Statistic value	Critical value		Statistic value	Critical value	
		10%	5%		10%	5%		10%	5%
r = 0	94.34*	88.75	97.78	53.30	99.23	108.3	146.75	170.30	181.95
r = 1	44.62*	40.23	44.89	25.67	47.52	52.49	78.30	83.79	90.23
r = 2	11.81	16.88	19.77	11.35	22.02	25.12	34.10	38.68	42.54
r = 3	0.95	3.63	5.06	4.77	8.37	10.18	12.81	13.92	16.03

(*) and (**) denotes rejection of the null hypothesis at 10% and 5% significance level.

Zero frequency

Basing on the asymptotic critical values we may conclude that at zero frequency at 5% significance level the number of cointegrating vectors equals to 2. On the contrary, using the small sample values at 5% level we cannot reject the null hypothesis that there aren't any cointegrating vectors. Thus at 10% significance level the test results show the presence of two cointegrating relations.

Greenslade *at al*. (2002) emphasise that the true number of cointegrating vectors lies between the numbers determined on the basis of the asymptotic and the small sample critical values. That is why, considering the test results, we assumed arbitrarily that in our model the rank of cointegration amounts to one. According to the *P-star* concept this relation should correspond to the long-run money demand function (8).

This decision has been supported by the analysis of the companion matrix in the section 4.3.2 where the number of common trends clearly indicates one cointegrating vector at zero frequency.

Biannual frequency

In case of biannual frequency the results of the trace test with asymptotic critical values at 5% significance level confirm the presence of one cointegrating vector. If we use the small sample critical values the null hypothesis of no cointegrating vectors cannot be rejected at both 5% and 10% levels.

This result means that the rank of cointegration at biannual frequency is zero or one. We will conduct further analysis under the assumption that the cointegration rank at this frequency equals one.

Annual frequency

At annual frequency the outcome of the trace test with asymptotic critical values suggests that the rank of cointegration amounts to 2 (at 5% level). On the other hand using small sample values we obtain the number of cointegrating vectors equal to zero (at 5% level). Considering the fact that the true cointegration rank lies between the numbers given by the asymptotic and the small sample critical values we conclude that the rank at annual frequency is 1.

4.3.4. The analysis of the cointegration space at zero frequency

Considering the arguments presented in section 4.3.3 we assumed that the rank of cointegration at zero frequency equals 1 and the obtained vectors represents the long-run money demand function expressed in (8). In the case when we have the only one cointegrating vector the system is exactly identified when we impose one restriction usually normalising this vector in respect to a certain variable (in the *P-star* model according to (8) it is the money variable). For that reason if we take the assumption of the long-run unit price elasticity of the money demand in (8), the model is overidentified and this assumption may be tested as proposed by Johansen and Schaumburg (1999).

Table 6 contains the parameter estimates of the cointegrating relation at zero frequency and the results of LR test of the long-run unit price elasticity of money demand in the *P-star* model.

Table 6. The cointegrating matrix at zero frequency

Variables	*M*3	*CPI*	*POT*	*SPREAD*	LR test statistic $\chi^2(1)$ (p-value)
Estimates/ Parameters	1	–1	–1.912 (0.114)	2.578 (0.390)	3.118* (0.077)
$\chi^2(1)$ (p-value)	12.05*** (0.001)	5.89** (0.015)	14.94*** (0.000)	12.32*** (0.000)	

(*), (**) and (***) denotes rejection of the null hypothesis at 10%, 5% and 1% significance level respectively.

Table 6 should be read as follows. In the first row the estimates (or fixed parameters) of the cointegrating relation followed by the asymptotic standard errors have been presented.[15]

The second row contains the value of appropriate statistic in the so called *exclusion test* (Johansen, 1995). The null hypothesis in this test assumes that the element of the cointegrating vector corresponding to a certain variable is equal to zero. According to the

[15] For *m*3 and *cpi* variables the parameters of the cointegrating relation have been imposed *a priori* according to (8).

alternative hypothesis this element (or elements in case of more than one cointegrating vectors) is different from zero. If the null hypothesis is true the test statistic will follow χ^2 distribution with one degree of freedom. Thus the large values of this statistic lead to rejection of the null hypothesis and confirm the presence of the variable in the cointegrating relation.

In our case the results of the test confirmed that all variables should appear in the cointegrating vector (interpreted as the money demand function).

The last column of the table 6 presents the value of the statistic in the above mentioned LR test of the overidentifying restrictions. The overidentifying restriction in the *P-star* model is the long-run unit price elasticity of money demand. At 5% significance level this restriction has not been rejected what means that the money demand function is a real money demand function and there is no money illusion in the Polish economy at zero frequency.

The signs and magnitudes of the parameter estimates in the cointegrating relation are in accordance with our expectations and the *P-star* concept. The demand for the real money is in the long-run an increasing function of the potential output and is the decreasing function of interest rate spread. Moreover, the value of money demand income elasticity exceeds unity (1.912) what may be explained as the presence of the wealth effect in the demand for money.

Beeby, Funke and Hall (1995) emphasise that the positive verification of the *P-star* theory depends not only on the presence of the stable cointegrating relation interpreted as the money demand function but also on the causality relationship between the real money gap and the price level. Barassi *et al*. (2001) show that testing for the long-run non-causality also matters in the sense of testing for weak exogeneity. That is because the long-run non-causality is necessary as well as sufficient for the long-run weak exogeneity of the variable with respect to the parameters of the cointegrating relations.

Considering the long-run causality relationship between the real money gap and the price level we will test a weak exogeneity of prices (*cpi*) with respect to the parameters of the long-run money demand equation (8). In case of a single cointegrating vector we must verify whether a particular element of the adjustment matrix is equal to zero. Taking into account that we have only one cointegrating vector in the system we may test the exogeneity of prices by comparing the particular parameter estimates in the adjustment matrix with the appropriate standard error (Barassi *et al*., 2001).

Table 7. The adjustment matrices at zero- and biannual frequency

	Zero frequency	Biannual frequency
$\Delta_4 M3$	–0.083 (0.085)	–1.657 (0.451)
$\Delta_4 CPI$	0.238 (0.037)	0.448 (0.184)
$\Delta_4 POT$	0.152 (0.050)	0.287 (0.215)
$\Delta_4 SPREAD$	–0.205 (0.059)	0.703 (0.316)

In the second column of the table 7 there are the estimates of elements of the adjustment matrix at zero frequency[16]. An estimate of the coefficient that measures the speed of adjustment in the prices equation (second row of the table 7) is significantly larger than its standard error. This means that in the analysed *P-star* model for the Polish economy the prices (*cpi*) are not exogenous with respect to parameters of the cointegrating relation what implies that between the real money gap and the price level exists the long-run causality relationship.

4.3.5. The analysis of the cointegration space at biannual frequency

At biannual frequency we assumed, similar to the zero frequency that the rank of cointegration is equal to one and that the single cointegrating vector represents the money demand equation.

In the table 8 (in the same scheme as in the table 6) the parameter estimates of the cointegrating relation and the results of the test of the overidentifying restrictions have been presented. However the results differ strongly from the results at zero frequency.

Table 8. The cointegrating matrix at biannual frequency

Variables	*M3*	*CPI*	*POT*	*SPREAD*	$cos(\pi t)$	LR statistic $\chi^2(1)$ (p-value)
Estimates / Parameters	1	–1	–0.399 (0.257)	–0.552 (0.329)	–0.005 (0.0042)	4.792** (0.029)
$\chi^2(1)$ (p-value)	10.33*** (0.000)	11.08*** (0.000)	1.24 (0.265)	2.82* (0.093)	-	

(*), (**) and (***) denotes rejection of the null hypothesis at 10%, 5% and 1% significance level respectively.

First of all at 5% significance level we must reject the hypothesis about the unit price elasticity of money demand, which implies the money illusion at biannual frequency. Thus the hypothesis cannot be rejected at 1% significance level.

[16] It should be kept in mind that due to the normalization of the cointegrating vector with respect to money and due to the unit price elasticity condition the adjustment coefficient in the prices equation should be positive.

Moreover, the test outcome implies that in the *P-star* model for the Polish economy the single cointegrating vector cannot be interpreted as the money demand equation (8). At 5% level the statistic values in the exclusion test (last row in the table 8) confirmed the presence of only two of four variables (*m*3 and *cpi*) in the cointegrating relation. Unfortunately in case of two other variables (*pot* and *SPREAD*) we cannot reject the null hypothesis that these variables should be excluded from the cointegrating space. For that reason the cointegrating relation at biannual frequency cannot be interpreted as the money demand function (8).

By interpreting the parameter estimates in the cointegrating relation we must emphasise that according to the assumptions of the *P-star* model the money demand is an increasing function of the potential output. The income elasticity of money demand amounts to 0.399, which is significantly below the unity. This value is in line with the Baumol-Tobin model where money has only a transaction function and the income elasticity equals to 0.5. It means that if there were no negative results of the exclusion test, which would have allowed interpreting the cointegrating relation at biannual frequency as the money demand function, the money demand equation at this frequency would have expressed mainly the transaction demand for money. On the contrary, according to the results described in section 4.3.4, at zero frequency the wealth effect has an important impact on the demand for money.

It is noteworthy that the estimate of the parameter corresponding to the interest rate spread is negative what is against the assumptions of the money demand theory described in section 2.2.

On the basis of the adjustment matrix estimate presented in the table 7 we may conclude that at biannual frequency money is not exogenous with respect to the real money gap.

4.3.6. The analysis of the cointegration space at annual frequency

Unlike two other frequencies, the cointegrating matrix at annual frequency has in the general case a complex form. Thus the economic interpretation of the parameters in the cointegrating relations is rather complicated. To simplify the error correction mechanism at this frequency in the first step we verified the hypothesis about the real cointegrating matrix at annual frequency (section 3.3). The results of the LR test indicate that at 5% significance level we must reject this hypothesis[17]. For that reason in further considerations we gave up to structuralize the cointegrating space at annual frequency as well as to find an economic interpretation for the cointegrating relations. Instead of that the table 9 contains the estimate

[17] The value of the test statistic 34.3 is larger than appropriate critical value from the chi2 distribution with three degrees of freedom what implies the rejecting of the null hypothesis.

of non-structural cointegrating matrix obtained directly by using the estimation algorithm proposed by Johansen and Schaumburg (1999).

Table 9. The cointegrating matrix at annual frequency

	*m*3	*cpi*	*pot*	*SPREAD*	*dummy variable*
β_R	1	–1.144	–1.528	–0.430	–0.022
β_I	1	–0.637	–0.116	–2.079	0.006

5. Summary of the results

1) The outcome of the research presented in the paper confirmed that in the Polish economy exists the long-run real money demand function where the demand for real money depends on the potential output and on the interest rate.
2) The real money gap, which expresses the difference between the actual real money and the theoretical level of real money derived on the basis of the long-run money demand equation may be the leading indicator for the inflation. This means that in the Polish economy exists the long-run equilibrium relationship between money and prices.
3) In case of zero frequency the prices variable is not weak exogenous with respect to the parameters of the money demand equation what confirms the direction of causality assumed in the *P-star* model (from money to prices). The error correction mechanism at zero frequency has been relatively strong – the estimate of the adjustment coefficient equalled to 0.238, which means that 2.6 quarters are required to close an initial disequilibrium (real money gap) by a half.
4) By separating the seasonal fluctuations from the long-run trend we were able to estimate the parameters of the long-run real money demand equation (at zero frequency). The estimation results confirmed that the demand for real money is an increasing function of the potential output and a decreasing function of the interest rate spread. The coefficients of the money demand equation are in line with the *P-star* concept and with the outcome of the analogical researches carried out for other countries.
5) The results of the trace test at biannual frequency suggest that a single cointegrating vector may exist. Unfortunately the outcome of the exclusion test indicates that only two of four variables are involved in the cointegrating relation. According to the test results two remaining variables (potential output and interest rate spread) shouldn't be included

in the cointegrating relation at biannual frequency and for that reason this relation cannot be interpreted as the money demand equation.

6) The statistic value in the trace test confirmed that at annual frequency the rank of cointegration equals one. Thus because of rejecting the hypothesis about the real cointegrating matrix we gave up to structuralize the cointegrating space and we left the cointegrating vectors without any economic interpretation.

Finally we want to emphasise that regarding to the small sample we were forced to make certain simplifying assumptions in the *P-star* model for the Polish economy. First we excluded all exogenous variables from the seasonal error correction model, which may have an impact on the inflation in the short-run. Secondly we involved an interest rate spread instead of two separate rates in the *P-star* model. We are aware that the annulations of these assumptions may have an impact on the final results.

Moreover, it must be kept in mind that the proposed seasonal cointegration approach bases on the assumptions, which in general hold only asymptotically. For that reason some properties of the estimators are only asymptotic what may have certain importance in the small sample.

That is why we should treat the final results with the proper precaution. Thus we think that it is worthy to repeat this research in the future using different simplifying assumptions, different estimation method (i.e. bootstrap) and a longer sample. This would help us to assess the sensitivity of the obtained results on the assumptions taken in this paper.

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