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Range-Dependent Utility

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Abstract

This paper introduces the concept of range-dependent utility. Instead of reference dependence which evaluates outcomes relative to some reference point, we postulate dependence on a given lottery (set of lotteries) outcomes range. In this way the decision maker is a fully rational expected utility maximizer only within a certain range. Range-dependent utility enables experimental results to be explained without recourse to the probability weighting function. Experimental data show that range-dependent utilities can be normalized to obtain decision utility - a single utility function able to describe decisions involving lotteries defined over different ranges. Both the data analysis as well as theoretical considerations concerning monotonicity indicate that the decision utility should be of S-shape.

Keywords: range-dependent utility, decision utility, Certainty Equivalent, quasilinear mean, Expected Utility Theory, Prospect Theory, Allais paradox, insurance and gambling

JEL Classification Numbers: D81, D03, C91

1 Introduction

It is widely believed that in order to explain some of the most prominent choice paradoxes of expected utility such as Allais type paradoxes, it is necessary to introduce some kind of probability weighting (evaluation function that is not linear in probabilities). We propose to adopt a different approach in which probability weighting is not necessary and yet some of the paradoxes and in particular Allais-type paradoxes may be explained. Our approach is based very closely on von Neumann and Morgenstern original idea of constructing utility indices, which is the crucial part of their proof of expected utility theory. Following Cox and Sadiraj (2006) we make a distinction between expected utility theories, which stands for all models based on a set of axioms among which there is independence axiom, and a specific expected utility model. We also follow Rubinstein (2006) (and Palacios-Huerta and Serrano (2006)) who claim that a lot of recent confusion around expected utility, which led some researchers to question it as a descriptive theory is caused by associating expected utility theory with the assumption of consequentialism[1]. Our model departs from the assumption of consequentialism but it does not depart from expected utility theory. In particular we do not question independence axiom - in our model independence axiom is satisfied but on a limited domain. The idea is the following. When people are making decisions they do not think about their lifetime wealth and all possible lotteries that are possible for them to play. They rather see at each time one or a few lotteries which are relevant for current decision. Each monetary lottery has its range of possible values (the interval between the lowest and the highest monetary consequence occurring with positive probability). We argue that when making decisions people mentally adapt to a certain range of possible values in a given lottery. Conceptually, it does not matter whether the decision maker perceives the relevant range of values as the interval between the lowest and the highest outcome in a given lottery or in a set of lotteries. In this paper we follow the extreme approach that the relevant range of values is defined by one lottery which is being considered only.

Psychologically speaking, range-dependent utility is based on the phenomenon of the mental perception system adapting itself to the range of outcomes under consideration (See Kontek (2011)). We will show that the model is capable of

[1] Consequentialism is based on the idea that there is a single preference relation over the set of lotteries with prizes being the "final wealth levels" such that the decision maker at any wealth level W who has vNM preference relation $\succsim_W$ over the set of "wealth changes" derives that preference from $\succsim$ by $L_1 \succsim_W L_2 \iff W + L_1 \succsim W + L_2$, where L_1 and L_2 are lotteries.

explaining certain behavioral phenomena that expected utility cannot explain and yet gives an alternative explanation to the existing explanations based on probability weighting - we shall focus here on three among the most important paradoxes of Expected Utility Theory, namely Common Ratio Effect, Common Consequence Effect (e.g. Allais paradox) and the coexistence of insurance and gambling.

Mathematically speaking, we will construct utility indices for consequences in a given lottery depending on the lottery range - the highest consequence in a given lottery being assigned the highest utility value and the lowest consequence in a given lottery being assigned the lowest utility value. This is what we will call range-dependent utility - utility that depends on a lottery range and hence is different for lotteries defined over different ranges. In terms of axioms we will show that the crucial expected utility axiom - independence - is satisfied only for lotteries defined over the same range, but may not be satisfied for lotteries defined over different ranges. This corresponds to weakening of expected utility model assuming consequentialism. As far as the proof of the first main theorem is concerned, the usual approach of following von Neumann and Morgenstern, i.e. adopting a binary preference relation on a set of lotteries as the primitive of the model, was discarded in favor of one that takes the lottery's certainty equivalent as the primitive of the model. This approach is based on de Finetti's famous quasilinear mean representation theorem, although the exposition in this paper adopts the classic treatment of Hardy et al. (1934). The de Finetti and von Neumann-Morgenstern theorems are very closely related and both of them lead to the Expected Utility representation. The main difference is that the former focuses on certainty equivalent rather than preference relation.

Range dependence should be distinguished from reference dependence, one of the crucial elements of Prospect Theory by Kahneman and Tversky (1979). In Prospect Theory a given outcome (for example a point representing *status quo*) is chosen as a reference point, and all the other outcomes are coded as gains or losses relative to this reference point. In range dependent utility model on the other hand there are two natural points which define a decision frame, namely the highest and the lowest outcome in a given lottery, with the interval between them being the lottery range.

We claim that range-dependent utility offers interesting alternative explanation to a couple of most important Expected Utility Paradoxes, and we believe that the concept has strong psychological motivation. However, in its basic form range-dependent utility model has two drawbacks. From the positive (descriptive) side, it has limited operational value, because in order to describe choices of a given individual we would have to elicit many utility functions, one for each lottery range. From the normative (prescriptive) side, range-dependent utility model has limited predictive power. Because we are allowed to use a separate utility function for lotteries defined over different ranges the model is very flexible accommodating new evidence, but it has so many degrees of freedom that it lacks strong testable predictions.

The two drawbacks, both on the positive as well as on the normative side of the model, can be fixed by imposing an additional assumption. Based on experimental data from Tversky and Kahneman (1992), we propose to adopt one shape valid for each lottery range. The idea is that the constructed utilities for different lottery ranges have similar shape when we normalize the lottery range to one interval. Mathematically speaking, if we assume that certainty equivalents for lotteries are scale and shifts invariant, we will get one shape for different range-dependent utilities - this is what we call decision utility. Decision utility solves the problems described above. There is now only one utility function shape and hence when we elicit it for a given decision maker we can use it for different lotteries later on. Moreover, the problem of too many degrees of freedom is also solved as now we have only one utility function shape for different ranges. Hence our model generates now strong testable predictions. Actually, we can risk a statement that decision utility model is now directly comparable with expected utility theory with consequentialism, since the only free "element" to choose in both models is the shape of utility function. On the other hand, Prospect Theory (both in its original as well as cumulative version) for example, has two free "elements" to choose, namely value function shape and probability weighting shape, hence Prospect Theory is not directly comparable in this sense with both decision utility model and expected utility with consequentialism[2].

Our results indicate that the decision utility should be of S-shape. This is confirmed both on the positive side by fitting curves into experimental data and by the potential to accommodate behavioral paradoxes such as Allais paradox as well as on the normative side by showing that an S-shape decision utility is consistent with monotonicity requirements with respect to First Order Stochastic Domination.

This paper is organized as follows. Range-dependent utility concept is in-

[2]Actually, one could argue that apart from value function shape and probability weighting function shape, there is yet another "free element" to choose in Prospect Theory, namely reference point, which can be adjusted to accommodate new evidence.

troduced in Section 2 using the original von Neumann and Morgenstern (1944) method of utility derivation and Tversky and Kahneman (1992) experimental data. The axiomatic treatment is then presented along with the representation result for range-dependent utility. The decision utility concept is presented in Section 3. Decision utility is a single normalized utility function that can be applied to lotteries having different ranges. Section 4 desribes how decision utility model accommodates experimental choice patterns in three Expected Utility Paradoxes, namely Common Ratio Effect, Common Consequence Effect (Allais type paradoxes) and the coexistence of insurance and gambling. Section 5 identifies conditions under which decision utility model produces results consistent with First Order Stochastic Dominance. Section 6 describes the estimation procedure used to obtain shapes of range-dependent utilities and decision utilities to match experimental evidence involving Certainty Equivalents. This estimation is also used to compare different models in terms of how well they predict the actual values for Certainty Equivalents in Tversky and Kahneman (1992) experiment. Section 7 presents a summary and a discussion on the similarities and differences between Expected Utility theories, Prospect Theories, and range-dependent and decision utility. Finally, appendices contain proofs of the main theorems and propositions as well tables with theoretical prediction of Certainty Equivalents in different models.

2 Range-dependent utility

2.1 vNM method of determining utility

In their seminal contribution von Neumann and Morgenstern (1944) proposed the following method of measuring utility: "Consider three events, C, B, A, for which the order of the individuals preferences is the one stated. Let p be a real number between 0 and 1, such that B is exactly equally desirable with a combined event consisting of a chance of probability $1-p$ for A and the remaining chance of probability p for C. Then we suggest the use of p as a numerical estimate for the ratio of the preference of B over A to that of C over A". Their suggestion may be presented as follows:

$$p = \frac{u(B) - u(A)}{u(C) - u(A)} \tag{1}$$

where u denotes the utility of an event. Equation (1) may be rearranged to give:

$$u(B) = (1-p)u(A) + pu(C) \tag{2}$$

Von Neumann and Morgenstern utility is cardinal and hence it is unique up to affine transformation. If we assign $u(A) = 0$ and $u(C) = 1$, we get:

$$p = u(B) \tag{3}$$

Note, that the utility of event B is expressed in terms of the probability of winning the lottery $(A, 1-p; C, p)$. It is further noted that event B is defined so that the decision maker is indifferent between this event and the lottery $(A, 1-p; C, p)$. We say that B is the certainty equivalent of lottery $(A, 1-p; C, p)$. We can rewrite equation (3) as:

$$p = u((A, 1-p; C, p)) \tag{4}$$

The probability of winning the lottery $(A, 1-p; C, p)$ thus determines directly the lottery utility. For instance, the utility of the lottery $(\$0, 0.7; \$100, 0.3)$ is simply 0.3. As a rule, the greater the probability of winning the lottery, the greater the lottery utility. By definition, the utility value belongs to the interval $[0, 1]$.

2.2 Experimentally determined range-dependent utilities

In what follows we will make use of the experimental data collected by Tversky and Kahneman (1992) involving elicited certainty equivalents for 28 lotteries for gains and 28 lotteries for losses. This experimental data played an important role in the motivation for Cumulative Prospect Theory. There were 7 pairs of lottery outcomes: $(\$0, \$50)$, $(\$0, \$100)$, $(\$0, \$200)$, $(\$0, \$400)$, $(\$50, \$100)$, $(\$50, \$150)$, and $(\$100, \$200)$. Certainty equivalents were collected for the prospects - please refer to Table 2 and 3 in Appendix 2. These (certainty equivalent, probability) pairs enable range-dependent utility (3) or (4) to be plotted for each lottery range being examined (see Figure 1). For example lottery $(\$0, 0.9; \$50, 0.1)$ has a Certainty Equivalent equal to \$9, which corresponds to the first to the left point in the upper graph in Figure 1. The plotted points then serve to estimate the range-dependent utility function within each range considered (for details describing estimation procedure please refer to Section 6.)

The plots for loss lotteries are not presented here, but the results are similar to those for gains. The range-dependent utility curves in respective ranges are not concave as usually assumed. Instead, they are S-shaped and are similar in all the ranges considered. The shape of the utility curve for a given outcome depends on the lottery range being considered. For instance, for the outcome

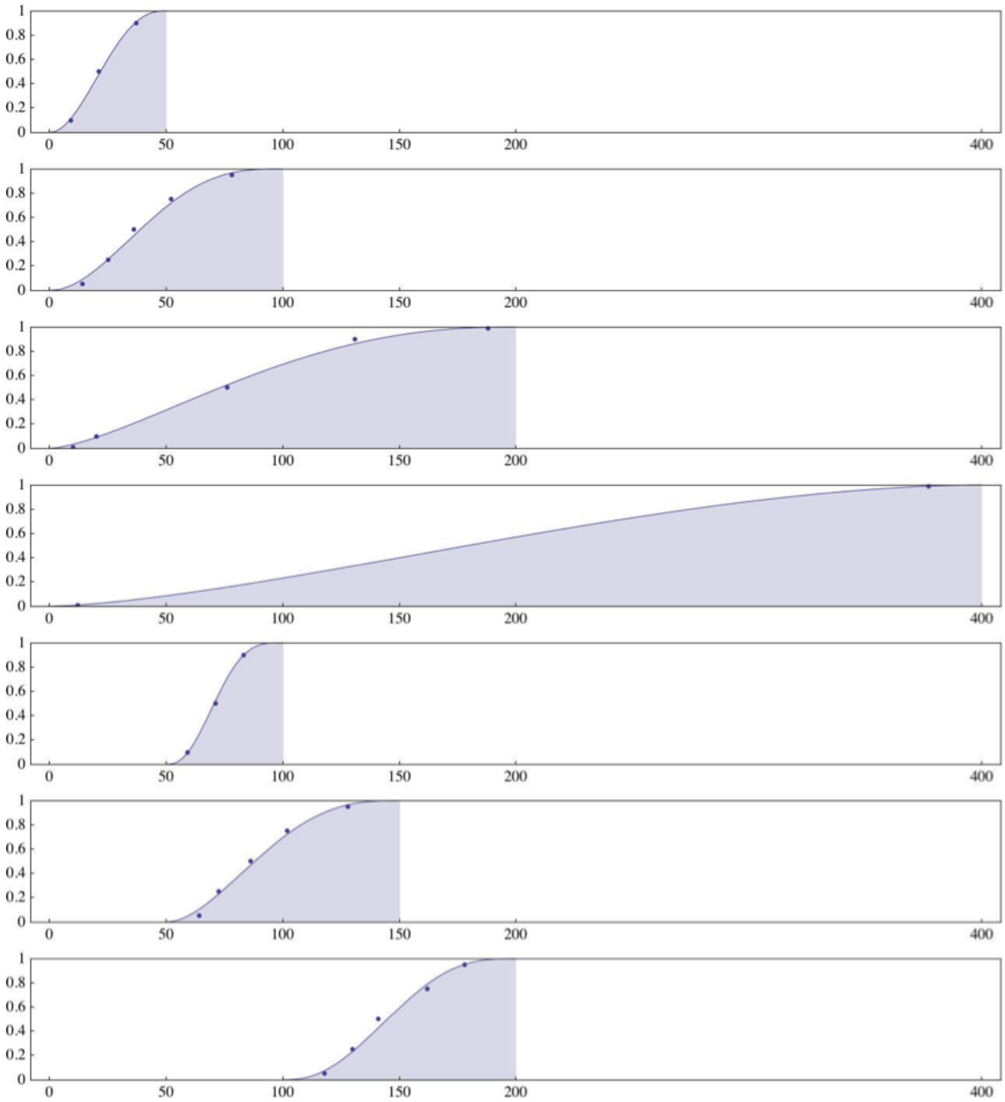

Figure 1: Range-dependent utilities obtained using data from Tversky and Kahneman (1992). Each plot corresponds with the respective lottery ranges: $[0, 50]$, $[0, 100]$, $[0, 200]$, $[0, 400]$, $[50, 100]$, $[50, 150]$, and $[100, 200]$. There are Certainty Equivalent values on the horizontal axis, and probability of getting a greater of the two prize on the vertical axis.

$75, utility is concave in the lottery range $[0, 100]$, roughly linear in the ranges $[0, 200]$ and $[50, 100]$, and convex in the ranges $[0, 400]$ and $[50, 150]$.

The shapes are analyzed in greater details on the left panel of Figure 2. It shows that the utilities in each range do not match each other and that it is impossible to determine a single utility that would describe the experiments in all the subranges with sufficient precision. The utility in the range $[0, 200]$ serves here as a pattern. Range-dependent utilities in the ranges $[0, 50]$,

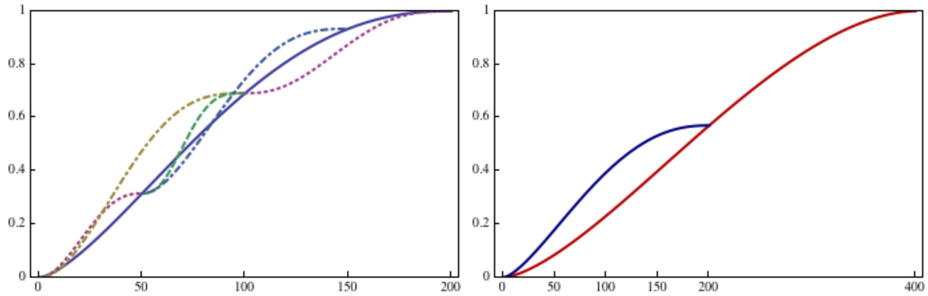

Figure 2: Range-dependent utility functions plotted on a single graph with proper scaling. On the horizontal axis there are Certainty Equivalent values and on the vertical axis there are probabilities or, alternatively, range-dependent utility values.

$[0,100]$, $[50,100]$, $[50,150]$, and $[100,200]$ are linearly transformed to have their endpoints located on the pattern utility. They do not match the shape of the pattern even in approximation. Additionaly, on the right panel of Figure 2 it is clearly the case that the utility in the range $[0,200]$ does not match the utility in the range $[0,400]$.

These results show that the respective range-dependent utilities accurately describe the experimental results in each lottery range. However, it is impossible to determine a single universal utility that would describe the experiments in all the sub-ranges (more details in section 6).

2.3 Axiomatic treatment

In what follows, the range-dependent utility theorem shall be stated. Both the formulation and the proof of this theorem makes heavy use of the statement and proof of the de Finetti theorem on quasilinear mean presented in Hardy et al. (1934).

Let X be a set of alternatives. The objects of choice are lotteries[3] with finite support:

$$L = \left\{ P : X \to [0,1] \,\middle|\, \#\{x | P(x) > 0\} < \infty, \ \sum_{x \in X} P(x) = 1 \right\} \tag{5}$$

[3]A note concerning notation: A lottery can be presented as a random-variable, as a density function for this random variable or as a cumulative distribution function of this random variable. In this subsection as well as in the proof of the theorem below we will use the convention that a lottery is denoted by its cumulative distribution function (not as a density function defined above). However in the other parts of the paper we use a different notation denoting a lottery as a random variable $\mathbf{x} = (x_1, p_1; x_2, p_2; ; x_n, p_n)$, where $p_i \geq 0$ for $i \in \{1, 2, ..., n\}$ and $\sum_{i=1}^{n} p_i = 1$. We switch between these notations mainly for the reasons of convenience as the three conventions are equivalent ways of representing lotteries.

Denote $\phi([x_l, x_u]) \subset L$ as the set of all simple lotteries defined over a finite interval $[x_l, x_u]$.

Axiom 1 (**Certainty**). *For a degenerate lottery $P \in L : \{P(x^*) = 1,\ x^* \in X\}$:*

$$CE(P) = x^*$$

Axiom 2 (**Within-Range Monotonicity**). *Fix range $[x_l, x_u]$. For every $P, Q \in \phi([x_l, x_u])$:*

$$P \text{ FOSD } Q \quad \Rightarrow \quad CE(P) > CE(Q)$$

Axiom 3 (**Within-Range Independence**). *Fix range $[x_l, x_u]$. For every $P, Q, R \in \phi([x_l, x_u])$ and $\alpha \in (0, 1)$*

$$CE(P) = CE(Q) \iff CE(\alpha P + (1 - \alpha)R) = CE(\alpha Q + (1 - \alpha)R)$$

Axiom 4 (**Reduction by Isolation**). *Consider a series $P^i \in L, i \in \{1, 2, ..., m\}$ of simple lotteries possibly having different ranges $[x_l^i, x_u^i]$. If for each simple lottery $P^i \in L$, there exists a single real number $CE(P^i)$ satisfying axioms 1-3, then the compound lottery:*

$$(P^1, q_1; P^2, q_2; ...; P^m, q_m)$$

is equivalent to a simple lottery

$$(CE(P^1), q_1; CE(P^2), q_2; ...; CE(P^m), q_m)$$

Theorem 1 (**Range-dependent utility**). *For every lottery $P \in L$ there exists a unique real number $CE(P)$ satisfying axioms 1-4 if and only if given any simple lottery $Q \in \phi([x_l, x_u])$ defined over a given range $[x_l, x_u]$ there exists a continuous and strictly increasing[4] function $u_{[x_l,x_u]}(\cdot)$, such that:*

$$u_{[x_l,x_u]}[CE(P)] = \mathbb{E}_{\mathbb{P}} u_{[x_l,x_u]}(x) \tag{6}$$

where expectation is taken with respect to probability distribution P.

Proof. In Appendix 1. □

It is necessary to digress briefly and discuss axioms 1-4. The theorem above is a slight modification of de Finetti theorem on quasilinear mean following Hardy et al. (1934) exposition. Certainty axiom is unchanged, whereas Within-Range Monotonicity and Within-Range Independence are modified versions of

[4]On the closed interval $[A, B]$.

Monotonicity and Independence axioms. The Reduction by Isolation axiom is new.

We refer to Axiom 1 as the Certainty axiom. This simply states that if the lottery is degenerate, i.e. it offers outcome x^* with certainty, then its certainty equivalent is equal to this certain outcome. Axiom 2 is called Within-Range Monotonicity axiom. It is satisfied for lotteries defined on the same range. It may be the case that for lotteries defined over different ranges, monotonicity is violated. This is analysed extensively in section 5. We find conditions which must hold so that monotonicity is satisfied for all lotteries, and not only for those which are are defined over the same range.

We shall refer to axiom 3 as the Within-Range Independence axiom. This states that the equality between certainty equivalents of two different lotteries defined over the same range should not be disrupted by the existence of a third alternative defined over the same range in a compound lottery. There are two things to note about this axiom. First, as the name correctly suggests, this axiom closely parallels the Independence axiom of von Neumann and Morgenstern. This is the heart of the Expected Utility representation. However, the version used here is the equality version as opposed to the weak inequality version usually adopted in the von Neumann and Morgenstern theorem. This means that the Within-Range Independence axiom adopted here is weaker than the corresponding Independence axiom of von Neumann and Morgenstern. This strengthens the present theorem. Second, the Within-Range Independence axiom is also weaker than the Independence axiom of von Neumann and Morgenstern because it only holds for same-range lotteries. The cost, however, is that the scope of the theorem is limited: only single, simple lotteries can be evaluated. For compound lotteries we need axiom 4 which is called Reduction by Isolation. The idea bears strong resemblance to the idea of isolation effect by Kahneman and Tversky (1979)[5]. The axiom assumes narrow framing. Given a compound lottery, the decision maker first evaluates each of the simple lotteries that compose it in isolation. In particular, she regards each simple lottery and its range separately without considering the other simple lotteries that make up the compound lottery. The axiom implies that the same

[5]"In order to simplify the choice between alternatives, people often disregard components that the alternatives share and focus on the components that distinguish them. This approach to choice problems may produce inconsistent preferences, because a pair of prospects can be decomposed into common and distinctive components in more than one way, and different decompositions sometimes lead to different preferences. We refer to this phenomenon as the isolation effect", Kahneman and Tversky (1979).

lottery may have different certainty equivalent when expressed as a multi-stage lottery and when expressed as equivalent one-shot problem.

The above theorem allows a lottery P (or in another notation $\mathbf{x}$) to be represented by its certainty equivalent $CE(\mathbf{x})$ via the range-dependent utility function $u_{[x_l,x_u]}(\cdot)$. This formulation abstracts from the notion of a preference relation. It is therefore a way of evaluating a given lottery rather than comparing two lotteries, as is the case with the von Neumann and Morgenstern theorem. Nevertheless, since a numerical representation of a lottery is obtained here, two lotteries can be compared on the basis of their certainty equivalents. Any lottery can be evaluated and its certainty equivalent found. Comparing two lotteries simply involves comparing their certainty equivalents.

3 Decision utility model

As noted in Section 2, the range-dependent utility curves presented in Figure 1 have similar shapes in different lottery ranges. This may lead to the conclusion that all range-dependent utilities can be described by a single normalized utility. All range-dependent utilities are normalized and all certainty equivalents CE are linearly transformed to the $[0, 1]$ interval using:

$$r = \frac{CE - x_l}{x_u - x_l} \tag{7}$$

All relative certainty equivalents r, together with their respective probabilities p, are presented on a single graph (see Figure 3). The normalized range-dependent utility is referred to as the decision utility function D (See Kontek (2011)). The function D was estimated using the Nonlinear Least Squares method with the assistance of the cumulative beta distribution function (i.e. regularized incomplete beta function). The respective values of the parameters α and β are 2.03 and 2.83 for gain prospects and 1.60 and 2.09 for loss prospects (Please refer to section 6 for details concerning estimation procedure).

3.1 The idea

To get some grasp of the idea of decision utility, lets compare it with range-dependent utility. Assume a lottery $\mathbf{x}$ with a range $[x_l, x_u]$. Then a range-dependent utility function:

$$u_{[x_l,x_u]}[x_l, x_u] \to [0, 1] \tag{8}$$

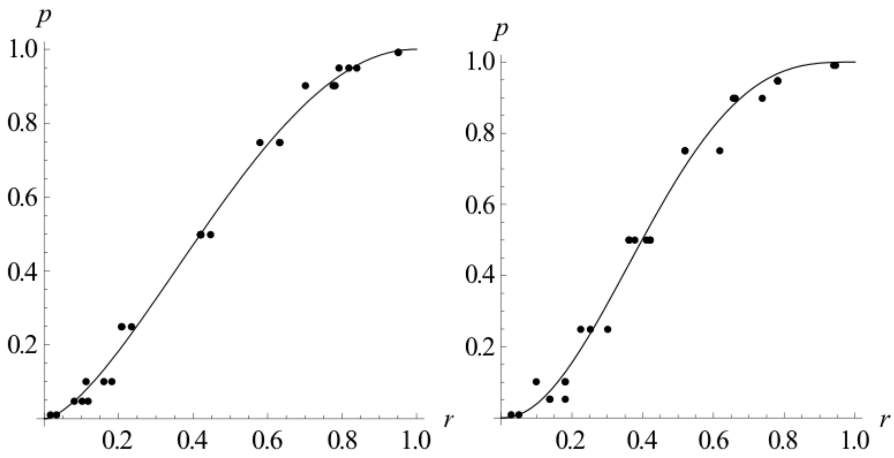

Figure 3: Tversky and Kahneman (1992) transformed experimental points and approximation $p = D(r)$ using the cumulative beta distribution function for loss prospects (left) and gain prospects (right). The horizontal axis represents respective ranges of monetary consequences normalized linearly into the interval $[0, 1]$. The vertical axis represents probability of getting x_u prize in each lottery.

normalized without loss of generality serves to determine the lottery certainty equivalent $CE(\mathbf{x})$ via the formula:

$$u_{[x_l,x_u]}[CE(\mathbf{x})] = \mathbb{E}u_{[x_l,x_u]}(\mathbf{x}) \tag{9}$$

This is the idea behind range-dependent utility. The idea behind decision utility on the other hand is as follows. Assume a normalized lottery $\mathbf{r}$ with relative outcomes:

$$r_i = \frac{x_i - x_l}{x_u - x_l} \tag{10}$$

Then a universal decision utility $D : [0, 1] \to [0, 1]$ is used to determine the normalized certainty equivalent of $\mathbf{r}$:

$$D[CE(\mathbf{r})] = \mathbb{E}_{\mathbf{r}} D(\mathbf{r}) \tag{11}$$

where $\mathbb{E}_{\mathbf{r}}$ is the expectation operator with respect to lottery $\mathbf{r}$. Finally, this certainty equivalent of $\mathbf{r}$ is denormalized to produce the certainty equivalent of $\mathbf{x}$:

$$CE(\mathbf{x}) = x_l + (x_u - x_l)CE(\mathbf{r}) \tag{12}$$

These 3 steps are presented graphically in Figure 4. It is necessary to follow this three step procedure as the universal decision utility may only be applied for normalized lotteries all having the same range of normalized outcomes.

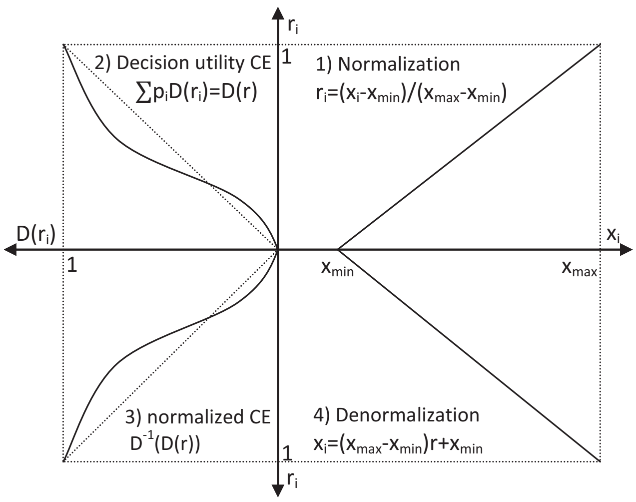

Figure 4: A three-step procedure to calculate decision utility.

3.2 The model

It follows from (11) and (12) that the lottery certainty equivalent is determined using the decision utility function D according to the single formula:

$$CE(\mathbf{x}) = x_l + (x_u - x_l)D^{-1}\left[\mathbb{E}D\left(\frac{\mathbf{x} - x_l}{x_u - x_l}\right)\right] \tag{13}$$

Note that D^{-1} denotes the inverse function of D, and $\mathbb{E}$ denotes expectation operator with respect to a distribution defined by lottery $\mathbf{x}$. Interestingly, x_l represents the riskless component of the lottery, and the difference $x_u - x_l$ represents the risky component, similarly as in the original Prospect Theory. We introduce the concept of equivalent probability p':

$$p' = \mathbb{E}D\left(\frac{\mathbf{x} - x_l}{x_u - x_l}\right) \tag{14}$$

The equivalent probability expresses the probability of winning the equivalent two-outcome lottery. Substituting a multi-outcome lottery by a two-outcome lottery results in a single p' value to describe the lottery, however many outcomes it has[6]. The value p' can be used to compare multi-outcome lotteries with the same outcome range, the lottery with the greatest equivalent probability p' is selected. However in order to compare lotteries defined over different ranges certainty equivalents have to be used in decision utility model. The

[6]This feature of the model has a very close parallel in the proof of von Neumann and Morgenstern Expected Utility Theorem, as the main idea there is, similarly, to successively replace complex (possibly compound) lotteries into a simple lottery involving only the maximal and the minimal outcome.

equivalent probability value cannot be used in this case contrary to Expected Utility Theory where there is only one utility function common to all ranges.

The case of two-outcome lotteries

For a two-outcome lottery, the equivalent probability (14) p' simplifies to the probability of winning the greater outcome:

$$p' = D(0)(1-p) + D(1)p = 0(1-p) + 1p = p \tag{15}$$

and the model given by (13) reduces to:

$$CE(\mathbf{x}) = x_l + (x_u - x_l)D^{-1}(p) \tag{16}$$

In this case the function $r = D^{-1}(p)$ maps lottery probabilities onto relative certainty equivalents. On the other hand a model with probability weighting can be written as[7]

$$CE(\mathbf{x}) = x_l + (x_u - x_l)w(p)$$

If $D^{-1}(p) = w(p)$, then the probability weighting model and the decision utility model are mathematically equivalent. This is however not the case anymore if we consider multi-outcome lotteries: in this case there is no direct mathematical translation of one model into another. It is also important to note, that even though there is a mathematical translation of one model into another, both models differ in the underlying psychological motivation.

3.3 Axiomatic treatment

We introduce additional axiom:

Axiom 5 (Shift and scale invariance). *Consider two lotteries $P, P' \in L$, such that P' is constructed from P by shifting all the lottery payoffs by $\delta \in \mathbb{R}$ and scaling all the lottery payoffs by $\gamma \in \mathbb{R}^{++}$. Then the following holds*[8]

$$CE(P') = \gamma CE(P) + \delta$$

[7] We can regard it as an evaluation function of Prospect Theory for two-outcome lotteries when we focus on the probability weighting aspect of Prospect Theory and leave the other elements aside. We can alternatively regard it as a version of Yaari (1987) dual theory for two-outcome lotteries.

[8] It will be convenient to write the above axiom in the following form:

$$CE(\alpha\mathbf{x} + \beta) = \alpha CE(\mathbf{x}) + \beta, \quad \forall\alpha > 0, \beta \in \mathbb{R}, \forall\mathbf{x}$$

Where $\mathbf{x}$ is another way to represent lotteries which belong to L.

This axiom says that certainty equivalent is invariant to scale and shift changes. It has two advantages. From the positive (descriptive) side it enables us to make range-dependent utility concept operationally meaningful. Now it is sufficient to elicit the decision utility function shape one time for a given decision maker and this shape may be used to evaluate different lotteries. With range-dependent utility this was not possible, as for lotteries with different ranges we would have to elicit a different utility function, which makes it operationally useless.

From the normative (prescriptive) side decision utility gives stronger testable predictions than range-dependent utility. By allowing strong testable predictions to be made, we can check whether these predictions are correct for a given choice behavior in different contexts. And when it gives incorrect predictions we can still compare it to other theories which give testable predictions and compare which theory does better in predicting behavior. We want to stress the fact that decision utility has only one free "element" to be adjusted in order to explain behavior of a single individual, that is the shape of decision utility. That makes it directly comparable with other expected utility models, for which it is also the case that the only free element to adjust is the shape of utility function. On the other hand, Prospect Theory has more than one free "element" to adjust and that is the shape of value function, but also probability weighting function[9]. Hence Prospect Theory should not be directly comparable with expected utility theories and decision utility model, because it has more degrees of freedom and as such is more flexible by definition.

The above axiom has one more interesting feature: within Expected Utility Theory assuming consequentialism the following is true[10]:

- CE is shift-invariant if and only if vNM utility function is CARA
- CE is scale-invariant if and only if vNM utility function is CRRA
- CE is shift- and scale-invariant if and only if vNM utility function is affine

So within Expected Utility Theory we can get maximum two of the following elements: risk aversion, CE shift-invariance, CE scale-invariance. On the other hand, within Decision Utility Model we can get all three at the same time.

Axiom 5 has also disadvantages because everything comes at a price. It is hardly a reasonable assumption to be made when we analyse ranges of lottery

[9]We could also count the determination of reference point as a free "element" which can be adjusted to explain behavior in Prospect Theory.

[10]See Pratt (1964) or Lewandowski (2011b).

values which are very different from each other. We should suspect that in the very high range of values the shape of decision utility function which describes attitudes towards risk should differ from the shape of decision utility function for very low ranges of lottery values. We should suspect that there is some wealth-effect similar to the effect of imposing DARA or more specifically CRRA utility class (the richer you get the less risk averse you become). Such wealth effect could be implemented for decision utility in the form of gradual (but regular) change of decision utility shape when going into higher ranges of lottery values. We believe however that it is a good starting point to model a benchmark case in which scale and shifts of consequences do not have impact on the shape of decision utility.

Below we present the theorem which gives axiomatic foundation for the decision utility model.

Theorem 2 (**Decision utility.**). *Axioms 1-5 are satisfied if and only if*

$$CE(\mathbf{x}) = x_l + (x_u - x_l)D^{-1}\left[\mathbb{E}D\left(\frac{\mathbf{x} - x_l}{x_u - x_l}\right)\right], \quad \forall x_u > x_l \tag{17}$$

Proof. In Appendix 1. □

4 Decision utility explanations of Expected Utility Theory paradoxes

The range-dependent/decision utility approach allows for an explanation of some of the most important Expected Utility Theory paradoxes without assuming probability weighting. Applying the decision utility function to the respective lottery ranges and determining the respective lottery certainty equivalents makes it possible to explain the effects which cannot be explained within Expected Utility Theory assuming consequentialism. In this section we will limit ourselves to three effects/paradoxes: Common Ratio Effect, Common Consequence Effect and the coexistence of insurance and gambling.

4.1 Common Ratio Effect

Lets begin with the Common Ratio Effect, that is presented here in the form analyzed by Kahneman and Tversky (1979).

Problem 1. *Choose between*

- *Lottery A:* \$4000 *with probability* 0.80 *and* \$0 *otherwise*
- *Lottery B:* \$3000 *with probability* 1.00

Problem 2. *Choose between*

- *Lottery C:* \$4000 *with probability* 0.20 *and* \$0 *otherwise*
- *Lottery D:* \$3000 *with probability* 0.25 *and* \$0 *otherwise*

Experimental results consistently reveal that most people choose option B in Problem 1 and option C in Problem 2. Expected Utility Theory, per contra, predicts that people would choose either (A and C) or (B and D), as the probabilities of winning the main prize in the second pair of choices differ by a common ratio factor of 4 compared with the first pair. Figure 5 presents this

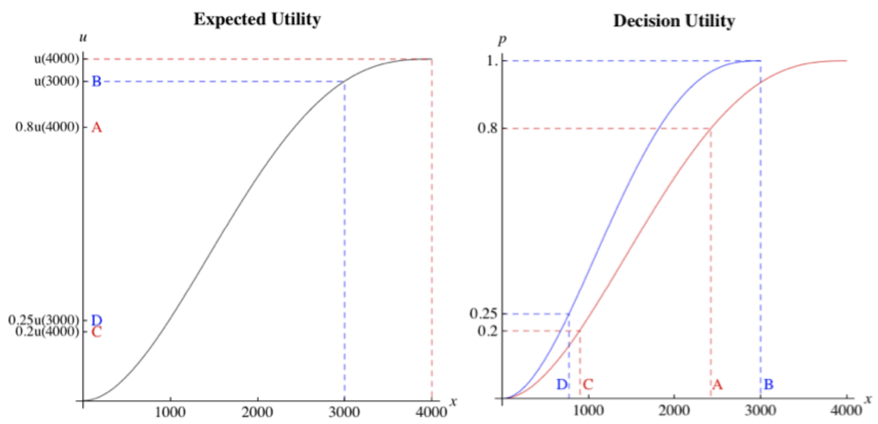

Figure 5: The Common Ratio Effect presented as a paradox when an absolute utility is applied (left) and explained using the decision utility function (right).

effect as a paradox when a hypothetical absolute utility is applied. No utility curve defined for outcomes expressed in absolute terms can explain this paradox. The solution to this problem presented in 5 (right) neatly demonstrates that a single decision utility applied separately to both lottery ranges predicts inconsistent choices. The shape of both utility curves is the same as they are rescaled decision utility functions. The interval of the red curve is $[0, 4000]$ as this corresponds with options A and C, which have a maximum outcome of \$4,000. The interval of the blue curve is $[0, 3000]$ as this corresponds with options B and D, which have a maximum outcome of \$3,000. The certainty equivalents of each option can be found on the x-axis. For greater probabilities, option B prevails over option A, but for lower probabilities, C is a better option than D. This is precisely what is observed in experiments.

There is another way to present the above argument and it is based on Marschak-Machina triangle (Machina (1987)). Figure 6 presents typical indifference curves in decision utility model with four lotteries A, B, C and D superimposed in the probability simplex.

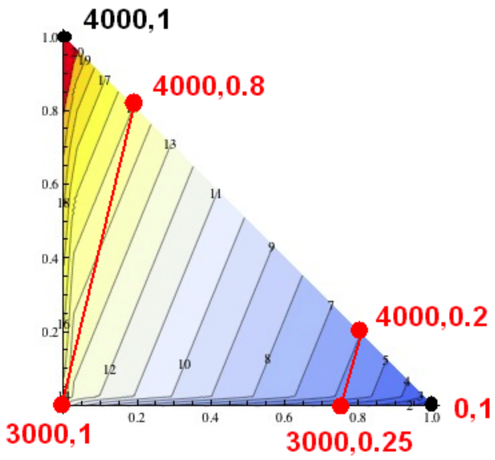

Figure 6: Typical decision utility model indifference curves in Marschak-Machina triangle: points represent lotteries used to illustrate Common Ratio Effect. Indifference curves are parallel lines except for the two legs of the triangle, where there is a discontinuity for each indifference curve. This corresponds to changing the range of the lottery.

The indifference curves are parallel lines as in Expected Utility Theory except for the edges corresponding to the two legs of the triangle. Going from the interior of the triangle into one of the legs results in the change of range[11]. This discontinuity makes it possible to accommodate the typical preference pattern regarding lotteries A, B, C and D. This "discontinuity in the legs" feature of decision utility is an alternative explanation to the fanning out hypothesis of Machina (1982).

4.2 Explanation of the Common Consequence Effect (Allais paradox)

The Allais Paradox is presented here in the form proposed by Kahneman and Tversky (1979).

Problem 3. *Choose between:*

[11]It is not true for the hypotenuse because going from the interior of the triangle into the hypotenuse results only in the vanishing of the intermediate lottery outcome which does not change the lottery range.

- *Lottery A:* $2,500 *with probability* 0.33*,* $2,400 *with probability* 0.66 *and* $0 *with probability* 0.01*.*
- *Lottery B:* $2,400 *with probability of* 1.00

Problem 4. *Choose between:*

- *Lottery C:* $2,500 *with probability* 0.33*,* $0 *with probability* 0.67
- *Lottery D:* $2,400 *with probability* 0.34*,* $0 *with probability* 0.66

The experiments show that most people choose option B in Problem 3 and option C in Problem 4, which contradicts the choices predicted by Expected Utility Theory (either (A and C) or (B and D)). The explanation of the paradox demonstrates the use of the decision utility model. The certainty equivalent of option A is determined using (13):

$$CE(A) = 2500D^{-1}\left[0.33D\left(\frac{2500}{2500}\right) + 0.66D\left(\frac{2400}{2500}\right) + 0.01D\left(\frac{0}{2500}\right)\right]$$

Note that, contrary to CPT, the order of outcomes evaluation in the formula is irrelevant. Using the parametrization cited earlier in this paper (See Figure 3), gives the following value:

$$CE(A) \approx 2500D^{-1}(0.98972) \approx 2184$$

Note, that the value of 0.98972 in the bracket represents the equivalent probability p' of winning the equivalent two-outcome lottery in the same range $[0, 2500]$. The remaining options B, C, and D have two-outcomes, so the simpler formula (16) applies:

$$\begin{aligned} CE(B) &= 2400D^{-1}(1) = 2400 \\ CE(C) &= 2500D^{-1}(0.33) \approx 766 \\ CE(D) &= 2400D^{-1}(0.34) \approx 750 \end{aligned}$$

It follows from the above that $CE(A) < CE(B)$ and $CE(C) > CE(D)$ which is in agreement with the experimental results.
More generally, we can easily find conditions which have to be satisfied in order that Common Consequence Effect be accommodated within decision utility model:

$$\frac{D^{-1}(0.33)}{D^{-1}(0.34)} > \frac{2400}{2500} > D^{-1}\left(\frac{0.33}{0.34}\right) \tag{18}$$

These two conditions require D to be sufficientl high for large arguments and sufficiently convex for small arguments. This suggests that Allais paradox is well accommodated when decision utility has an S-shape.

4.3 Insurance and gambling coexistence in the decision utility model

It is widely known that one decision maker is likely to buy insurance and and the same time may go to a casino to gamble. Expected Utility theory assuming consequentialism cannot accommodate such evidence. Kahneman and Tversky (1979) suggested that gambling and insurance may coexist due to overweighting of small probabilities. Decision utility model offers an alternative explanation which does not require probability weighting and it is based solely on the shape of the utility function. Our explanation is similar to the explanation postulated by Markowitz (1952) which involves a double S utility function for gains and losses. Figure 7 shows a typical decision utility shape. Now consider problems

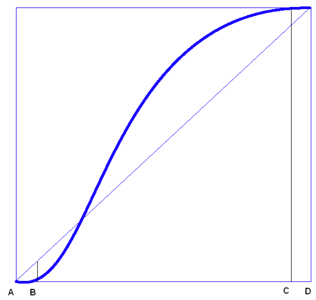

Figure 7: The shape of decision utility: this shape and dependence on the lottery range make it possible to accommodate the coexistence of gambling and insurance.

of fair insurance and of fair gambling. Let's take a simple model of fair insurance of a house (the value of a house is H and the probability it gets totally destroyed is p and is very small) in which there are only three possible outcomes[12]:

- A house gets destroyed and it was not insured (a value of 0)
- A house stays as it was and it was not insured (a value of H)
- The house was insured, and the decision maker pays the premium for it - so in case the house gets destroyed the decision maker receives a full compensation (a value of $H - pH$)

Notice that since the insurance premium costs pH it is a fair insurance problem. Since p is a small positive number, the three consequences above may be

[12]The decision utility model assumes scale and shift invariance (Axiom 5) and hence we are free to choose a unit and a zero for the values of each monetary consequence. It also means that we abstract here from the issue of what is to be regarded as gains and what is to be regarded as losses (reference dependence). We believe that it is a very important issue, but we don't want to deal with it here.

assigned values A, D and C, respectively in Figure 7. If the decision maker buys an insurance, then his decision utility value can be read directly for argument C. If the decision maker does not buy an insurance, his expected decision utility lies on the straight line connecting the two extreme ends of the decision utility function (corresponding to points A and D), exactly below the decision utility value for point C. Since in this region the decision utility values happens to be above the straight line, the decision maker will prefer to insure his house.

Now imagine a different scenario. A fair gambling problem in which you can win P with probability p, where p is a very small number, is considered where there are three possible consequences:

- A lottery ticket is bought but there is no prize (a value of $-pP$)[13]
- A lottery ticket is bought and the main prize is won (a value of $P - pP$)
- A lottery ticket is not bought so we don't have to pay for it (a value of 0)

Notice that since the price for the lottery tickets is pP, this is a fair gambling situation. Now since p is a very small number we can assign each of the above consequences values in Figure 7, the values for the following arguments: A, D and B, respectively. If the decision maker does not buy lottery ticket her decision utility value may be read directly for the argument in point B. If the decision maker buys lottery ticket then her expected decision utility value lies on the straight line connecting the two extreme ends of the decision utility function (corresponding to points A and D), directly above point B. Since in this region decision utility function values are below the straight line, the decision maker will choose to gamble.

The two scenarios above illustrate that in decision utility model the coexistence of gambling and insurance may be natural and it is caused by the S-shape of the decision utility together with range-dependence. Due to the assumption of scale and shifts invariance for certainty equivalents, we are able to abstract from the issue of reference dependence and what is to regarded as gains and losses and to concentrate on the issue of range-dependence which lies in the heart of the proposed model.

[13] As before we are free to choose a unit and a zero for the monetary consequences. However the values assigned in the problem of gambling should not be compared with the values assigned in the problem of insurance.

5 Monotonicity properties of the decision utility model.

Decision utility model has one objectionable property, namely that monotonicity with respect to First Order Stochastic Domination holds only for lotteries defined on the same range. In what follows we will identify conditions under which monotonicity is satisfied also for lotteries defined over different ranges.

Before we turn into a proper mathematical treatment, it is instructive to present a simple intuition for the kind of result we are seeking for. Let's take the following 3-outcome lottery $L \equiv (x_1, p_1; x_2, p_2; x_3, p_3)$. Let's define η, the relative position of a "x_2" outcome and r, the relative certainty equivalent as:

$$\eta = \frac{x_2 - x_1}{x_3 - x_1}, \quad r: \; D(r) = p_3 + p_2 D(\eta)$$

Now we obtain the formula for certainty equivalent of L within decision utility model:

$$CE(L) = x_1 + (x_3 - x_1)r \tag{19}$$

To find conditions under which the decision utility model does not exhibit monotonicity violations we need make sure that increasing x_1 or x_3 does not make the resulting lottery worse than the original one. In order to check it, we have to analyze the following derivatives.

$$\begin{aligned} \frac{dCE(L)}{dx_1} &= (1-r) - p_2 \frac{(1-\eta)D'(\eta)}{D'(r)} \\ \frac{dCE(L)}{dx_3} &= r - p_2 \frac{\eta D'(\eta)}{D'(r)} \end{aligned}$$

We want to make sure that the above derivatives are non-negative. Intuitively speaking and leaving the technical details aside for a moment, for the first derivative above we require that when η tends to 0, $D'(\eta)$ should be sufficiently small, and for the second derivative above we require that when η tends to 1, $D'(\eta)$ should be sufficiently small. In order to simultaneously satisfy these two conditions, it seems a natural choice to select an S-shape decision utility, since it is precisely the case for this shape that the derivative is small both at the left and at the right endpoint. What is immediately apparent is that an inverse S-shape function is on the other extreme, since it has derivatives of large values at both left and right endpoints. The first conclusion therefore, is that it is easiest to satisfy monotonicity if the decision utility function is of an S-shape. Along with the positive arguments of fitting curves to experimental data in sections 2 and 3 as well as the potential to accommodate behavioral phenomena analyzed in section 4, this is yet another reason (on the normative side) that supports an S-shape decision utility hypothesis.

After brief intuitive explanation, we are now ready to present a proper mathematical treatment of the issue of monotonicity. We will first define two measures of risk aversion in the context of decision utility. It is important to distinguish them from the similar concepts existing in traditional models where utility is defined over wider set of monetary outcomes. Here utility is defined meaningfully only over the respective lottery range, which has important implications also for the underlying measures of risk aversion.

Let's define measures of Relative Risk Aversion and Inverse Relative Risk Aversion for decision utility as:

$$\begin{aligned} RRA(\eta) &= -\frac{\eta D''(\eta)}{D'(\eta)} \\ InvRRA(\eta) &= \frac{(1-\eta)D''(\eta)}{D'(\eta)} \end{aligned}$$

where $\eta \in [0,1]$ is a normalized monetary outcome (the relative position of a given monetary outcome in the lottery range). Observe that the two notions, the former one existing in the Expected Utility Theory as well, and the latter one being a new concept specific to decision utility, are related to each other by the following expression:

$$InvRRA(\eta) = RRA(\eta) - \mathrm{ARA}(\eta)$$

where $\mathrm{ARA}(\eta)$ is a measure of absolute risk aversion defined as in Expected Utility Theory, but over a different domain.

Now we state the main proposition.

Proposition 5.1. *Sufficient conditions for monotonicity in decision utility model is that for all $\eta \in [0,1]$, Relative Risk Aversion is non-decreasing and Inverse Relative Risk Aversion is non-increasing.*

Proof. In appendix 1. □

This proposition translates the intuition presented earlier in the section into precise mathematical statement which makes use of familiar concepts of risk aversion. Using this Proposition we can now verify whether a specific functional form of decision utility satisfies or violates model monotonicity. In this paper we examine some basic functional forms. First of all, the limiting functions having constant RRA and InvRRA can be stated as solutions to simple differentiable equations:

$$\begin{aligned} RRA(\eta) &= \text{const} \iff D_1(\eta) = \eta^c, \quad c > 0. \\ InvRRA(\eta) &= \text{const} \iff D_2(\eta) = 1-(1-\eta)^d, \quad d > 0. \end{aligned} \tag{20}$$

Relative Risk Aversion and Inverse Relative Risk Aversion values for the limiting functions are presented in the following table: Notice that in case of

	$D_1(\eta)$	$D_2(\eta)$
$RRA(\eta)$	$1-c$	$\frac{\eta}{1-\eta}(d-1)$
$InvRRA(\eta)$	$\frac{1-\eta}{\eta}(c-1)$	$1-d$

Table 1: Relative Risk Aversion and Inverse Relative Risk Aversion values for the limiting functions.

the power function D_1, $RRA(\eta)$ is constant in its argument for every value of parameter c and $InvRRA(\eta)$ is decreasing in its argument for $c > 1$, i.e. when the curve is convex. In this case (and not in the opposite case when $c < 1$) monotonicity is satisfied. Similarly, in case of the inverse power function D_2, $InvRRA(\eta)$ is constant in its argument for every value of parameter d and $RRA(\eta)$ is increasing in its argument for $d > 1$, i.e. when the curve is concave. In this case (and not in the opposite case when $d < 1$) monotonicity is satisfied.

The two limiting functions assume either convex or concave shape so they are not suitable for describing more complex convex-concave shapes typically required for decision utility. Therefore it is instructive to examine Cumulative Two-Sided Power Distribution CTSPD (see Kotz S. (2004)), which is defined using power function for smaller arguments, and inverse power function for larger arguments:

$$\begin{aligned} D(\eta) &= p_0\left(\frac{\eta}{\eta_0}\right)^{\frac{\alpha\eta_0}{p_0}} && \text{for } 0 \le \eta \le \eta_0 \\ D(\eta) &= 1-(1-p_0)\left(\frac{1-\eta}{1-\eta_0}\right)^{\frac{\alpha(1-\eta_0)}{1-p_0}} && \text{for } \eta_0 \le \eta \le 1 \end{aligned} \tag{21}$$

The point (η_0, p_0) is the inflection point of the curve. The shape of the CTSPD is presented in Figure 5 (a blue segment for smaller arguments, and a red segment for larger ones).

The requirement for non-decreasing Relative Risk Aversion can be translated graphically into shapes located right to the power function. The requirement of non-increasing Inverse Relative Risk Aversion can be translated graphically into shapes located left to the inverse power function. As can be

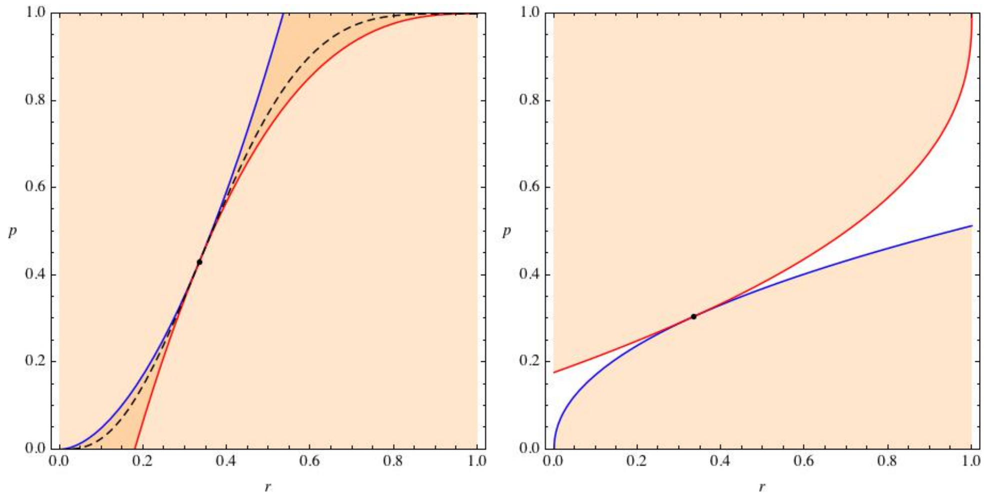

Figure 8: Shapes of the Cumulative Two-Sided Power Distribution. A blue segment describes the function for arguments below the inflection point (marked as a dot); a red segment describes the function above the inflection point. The area of shapes satisfying both monotonicity conditions is marked dark orange, the area of shapes satisfying only one monotonicity condition is marked light orange, and the area of shapes satisfying no conditions is marked white. A Cumulative Beta Distribution with parameters $\alpha = 3$ and $\beta = 5$ is presented in the left panel (dashed line) as an example of shape satisfying both monotonicity conditions.

seen both monotonicity conditions are satisfied only for S-shaped decision utility curves. An example of such a curve is demonstrated in the left panel as a dashed line. Very clearly CTSPD itself satisfies both conditions in this case as well. However, no curve satisfying both conditions can be placed in the right panel. This confirms that monotonicity is violated for inverse S-shaped decision utility functions.

6 Estimation technique

Estimation in this paper is important due to the following reasons:

- To obtain shapes of range-dependent and decision utilities based on experimental data from Tversky and Kahneman (1992).
- To make prediction of Certainty Equivalent values on the basis of these shapes.
- To compare range-dependent utility and decision utility prediction with prediction given by the most prominent models of decision-making un-

der risk, namely traditional Expected Utility (assuming consequentialism) and Prospect Theory.

The shapes of different utility function were presented before in sections 2 and 3. The prediction on the other hand as well as the comparison between different models is presented in Appendix 2 in Table 2 for prospects involving gains and in Table 3 for prospects involving losses. In what follows we shall briefly sketch the estimation procedure used in these tasks:

Estimation was performed using paramateric Nonlinear Least Squares method with help of the cumulative beta distribution function (regularized incomplete beta function) with two parameters. This function was chosen because it gives a lot of flexibility regarding utility function shapes. Moreover, in cases where it was necessary, we made additional checks using different parametric families, but the results were not much different from the current ones.

For each range, the endogenous variable was data on median Certainty Equivalents obtained from a group of 25 subjects in Tversky and Kahneman (1992) experiment. The exogenous variable was probability of getting first prize xu in a given pair in case of range-dependent utilities and additionally data concerning lottery outcomes in case of other models. In case of estimating range-dependent utilities, where for each range there were only several available data points, we also assume additional data points: Certainty Equivalents for two "endpoint" lotteries $(x_l, 1; x_u, 0)$ and $(x_l, 0; x_u, 1)$, their values being - by definition - equal to x_l and x_u, respectively. These points "fix" the utility curves at their endpoints. We are aware that, still, the number of observations used in estimation of range-dependent utilities is small. The important fact however is that there is a clear pattern visible in each range: the range-dependent utility is convex for values close to the lower range endpoint (risk-seeking) and concave for values close to the upper range endpoint (risk aversion).

Decision utility model was estimated using two methods:

- DU1 - All Certainty Equivalents are first normalized to the range $[0, 1]$ and then used to estimate the decision utility function. The resulting decision utility function is then applied in the model (13),
- DU2 - The model (13) is directly estimated.

We also estimated the Expected Utility model with one universal function over the whole $[0, 400]$ range[14]. Abstracting from the goodness of fit, it is interesting

[14]The estimated function was also a cumulative beta function.

that the best fitted function turns out to be globally concave as usually assumed by the traditional Expected Utility model.

It is instructive to compare predictions based on the different estimated models - see Table 2 for prospects involving gains and Table 3 for prospects involving losses in Appendix 2. We calculated the total sum of squared errors for predictions offered by Expected Utility and range-dependent utilities. We obtained a value which is more than 10 times higher in case of Expected Utility (4097 versus 338). For loss lotteries the difference is even bigger (1841 versus 71). The comparison with range-dependent utility is included here as a useful benchmark.

In case of decision utility, the sum of squared errors of the prediction is about three/four times smaller than for the Expected Utility model and prospects involving gains (value 1402 for DU1 and 929 for DU2 against 4097 for EU model). For loss lotteries the differences is even larger (323 for DU1 or 211 for DU2 against 1841 for EU model). Apart from the comparison involving the Sum of Squared Errors, it is instructive to compare the two models instance-wise. Decision utility DU1 gave better prediction than Expected Utility model for 24 out of 28 prospects, both for prospects involving gains as well as for prospects involving losses.

It is fair to compare decision utility directly with Expected Utility (with consequentialism), as in the two models the only free element to choose is the shape of ONE utility function. It turns out that decision utility gives much better predictions than Expected Utility. This should also be compared with the graphical representation demonstrating impossibility of fitting one range-indepenent function that matches the experimental evidence - see Figure 2.

Finally, in Table 2 and 3 we included a comparison of predictions using decision utility model and Prospect Theory model. Prospect Theory predictions were obtained on the basis of parameters estimated by Tversky and Kahneman (1992). For prospects involving gains, Prospect Theory does slightly better (SSE is equal to 657 for PT as compared to 1402 for DU1 and 929 for DU2), but for prospects involving losses, the situation is reversed (SSE is equal to 354 for PT and 323 for DU1 and 211 for DU2). Apart from the comparison involving the Sum of Squared Errors, it is instructive to compare the two models instance-wise. Decision utility DU1 gave better prediction than Prospect Theory PT for 9 out of 28 cases for prospects involving gains and for 17 out of 28 cases for prospects involving losses.

7 Discussion

The idea of range-dependent utility introduced in this paper needs to be compared with the two most commonly discussed models in decision theory under risk, namely Expected Utility theories and Cumulative Prospect Theory.

There is a lot of confusion in the profession whenever economists talk about Expected Utility Theory. The reason for this is that the term Expected Utility Theory means different things to different people. To clarify, we shall use the term Expected Utility Theory when we discuss the von Neumann Morgenstern theorem or, alternatively, the de Finetti quasilinear mean theorem. These theorems are silent about the identity of lottery prizes (i.e. what is to be regarded as lottery prizes) and they do not specify whether there exists a single preference relation that describes all decisions under risk for a single individual or whether there exists a series of preference relations each representing the individuals decisions in a different decision context. When economists argue for or against Expected Utility, they usually have a specific version of Expected Utility in mind[15], viz. Expected Utility Theory with the additional assumption of consequentialism. Consequentialism (see Rubinstein, 2009) assumes that lottery prizes represent the final wealth positions of the decision maker in a given risky situation. It further assumes that there exists a single preference relation for all the risky decisions of an individual. It is important to observe that consequentialism has serious consequences but is not part of the von Neumann and Morgenstern theory.

Other possible assumptions can be made instead of consequentialism. If lottery prizes are assumed not to be final wealth positions but changes relative to some reference wealth, then the concept of reference dependence arises. These changes may be nominal as is the case of reference dependence in Prospect Theory or they may be relative to some initial wealth (See Lewandowski (2011a)). Furthermore, instead of assuming a single preference relation for all the risky decisions of an individual, it is possible to assume different preference relations regarding risky decisions in different decision contexts. For example, one preference relation for an investment banker when he invests his companys money, another one when he invests his own money, and yet another when he gambles in a casino (See Lewandowski (2011a)). Similarly, a different preference relation can be assumed for choices regarding lotteries defined over different ranges. This is the idea behind range-dependent utility that the Expected Utility ax-

[15]See Cox and Sadiraj (2006).

ioms hold for lotteries defined over the same range. This is a special kind of narrow framing in which the decision maker is only fully rational when choosing between lotteries defined over the same range.

Range-dependent utility can explain many paradoxes that Expected Utility (assuming consequentialism) cannot. However, range-dependent utility in its basic form has two potential drawbacks.

From the positive (descriptive) side, it has limited operational value, because in order to describe choices of a given individual we would have to elicit many utility functions, one for each lottery range. The beauty of expected utility model assuming consequentialism is that it is sufficient to elicit a utility function for a given decision maker once and, provided it does not change over time, use it to evaluate any new lottery. This is a drawback on the positive (descriptive) side of the range-dependent utility model.

Second drawback is on the normative (prescriptive) side. Range-dependent utility is very flexible because it allows different utility shapes for lotteries defined over different ranges. However it also implies that its predictive power is limited, as it has many degrees of freedom. In other words, it lacks strong testable predictions which is what we should expect from a good theory. Expected utility theory assuming consequentialism may sometimes give wrong predictions, but it certainly gives strong testable predictions and we can judge the quality of these predictions. A theory which does not give testable predictions and is so flexible that it is capable of accommodating almost any new evidence that occurs, is not a good theory - in particular not in normative sense, due to overfitting: by explaining too much of the individual peculiarities in the data, we miss the general laws underlying the decisions and which we are aiming at.

The two drawbacks of range-dependent utility described above can be fixed by an additional assumption. Based on experimental data of Tversky and Kahneman (1992) we introduced an assumption of scale and shift invariance for certainty equivalents. Decision utility assumes that the utility function is still defined over a lotterys range, but when two lotteries are defined over different ranges and these ranges are normalized to the common interval $[0, 1]$, then the shape of the normalized range-dependent utility for one lottery is the same as the shape of the normalized range-dependent utility for the other. This common utility function is called decision utility. The decision utility model thus restores the universality of the Expected Utility (assuming consequentialism), while at the same time enabling certain behavioral phenomena to be explained.

Expected Utility (assuming consequentialism) thus assumes that the Expected Utility axioms are satisfied for all lotteries (i.e. the decision maker is fully rational) whereas range-dependent utility and decision utility models assume the Expected Utility axioms to be satisfied only for lotteries defined over the same range. However, both the Expected Utility (assuming consequentialism) and decision utility models have a single utility function for a given individual. It is therefore reasonable to check which model performs better empirically (i.e. which one gives better predictions).

On the other hand, Prospect Theory and Cumulative Prospect Theory differ from the range-dependent utility and decision utility models because they both weight probabilities, whereas the latter treat probabilities linearly as in Expected Utility Theory. Probability weighting was introduced in the early 1950s to describe behavioral patterns actually observed in lottery experiments which could not be explained by Expected Utility Theory (assuming consequentialism). The introduction of this concept coincided with the early psychological research on probability perception by Edwards (1954) and with the theoretical work on subjective probability by Savage (1954). This indicated that probability weighting was the valid way to explain irrational behaviors, where irrational meant not conforming to the Expected Utility axioms of rationality. The results of the present paper suggest that, provided the issue of reference dependence is put aside, the two approaches, namely probability weighting and the decision utility model may be used alternatively to explain experimental behavior in case of two-outcome lotteries. This is to say that they can accommodate the same experimental evidence although both approaches differ in psychological motivation. Furthermore, Yaaris (1987) dual theory, which requires cumulative probability weighting but assumes a linear utility function, is exactly the same for two-outcome lotteries as the decision utility model in that both models can accommodate the same experimental evidence. The equivalence between the decision utility and probability weighting (both individual and cumulative) models is clearly no longer true for multi-outcome lotteries. The spectrum of behavioral patterns in this case differs for the two models allowing for easier model discrimination in specific data analyzes.

Although decision utility and probability weighting can accommodate the same experimental data in certain cases, adopting one approach over the other has profound consequences for the assumed principles underlying human behavior. Whereas probability weighting excludes rationality as defined by the von Neumann Morgenstern axioms completely, the range-dependent/decision utility approach assumes that people are rational when considering lotteries defined over the same range. It is only when considering lotteries defined over different ranges that people may behave irrationally. Second, in the case of multi-outcome lotteries, the original Prospect Theory, which was based on psychological evidence, runs into trouble because of the existence of possible first-order stochastic dominance violations caused by probability weighting. Disposing of this undesirable feature by adopting cumulative probability weighting (Cumulative Prospect Theory) is mathematically elegant, but psychologically less founded than the original version. On the other hand, range-dependent/decision utility with an S-shape does not encounter problems with first-order stochastic dominance. Moreover, it is well grounded in psychology, being based on the phenomenon of the mental perception system adapting itself to the range of outcomes under consideration (See Kontek (2011)).

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Appendix 1

Proof of theorem 1

This subsection contains the proof of main theorem concerning range-dependent utility model (Theorem 1). It is based closely on Hardy et al. (1934) exposition of the de Finetti quasilinear mean representation theorem. Before we begin the proof, we will make a couple of preliminary observations.

Obs. 1 Consider the following step function:

$$\beta(x) = \frac{1}{2}(1 + \text{sgn}(x))$$

It has the single jump 1 at $x = 0$. Define another function:

$$\beta_\alpha(x) = \beta(x - \alpha)$$

Then we can write the CDF which describes a given lottery $P \in L$ as:

$$P(x) = \sum_{i=1}^{n} p_i \beta_{x_i}(x)$$

This way of representing a lottery will be used in the proof.

Obs. 2 Another observation which we will make prior to proving the theorem is the following:

Lemma 1. *Axiom 3. above implies the following:*

$$CE(q_1 P^1 + q_2 P^2 + q_m P^m) = CE(q_1 Q^1 + q_2 Q^2 + q_m Q^m)$$

if $CE(P^i) = CE(Q^i)$, *for all* $i \in \{1, 2, ..., m\}$ *and* $\sum_{i=1}^{m} q_i = 1$.

Proof. Use axiom 3. twice to obtain $CE(\alpha P^1 + (1-\alpha)P^2) = CE(\alpha Q^1 + (1-\alpha)Q^2)$, $\forall \alpha \in [0, 1]$, whenever $CE(P^1) = CE(Q^1)$ and $CE(P^2) = CE(Q^2)$. The result follows by using this argument repeatedly. □

Now we can proceed to the proof of the main theorem:

Proof. The proof will be broken down into three steps:

Step 1 The converse part of the theorem is: if $CE(P)$ is defined by (19), then it satisfies axioms 1., 2. and 3.

It is obvious that $CE(P)$ satisfies axiom 1. Now suppose that P^1 FOSD

P^2, for P^1 and P^2 defined on the same range. Then there is a positive number b and an interval (a, c) such that

$$P^1(x) > P^2(x) + b > P^2(x)$$

for $x \in (a, c)$. Instead of writing $u_{[x_l,x_u]}(x)$ as in the statement of the theorem, we shall use a shorthand notation $u(x)$, bearing in mind that utility function is range-dependent: here both lotteries P_1 and P_2 are defined on the same range.
Using intergration by parts, we have:

$$\begin{aligned} u(CE(P^2)) - u(CE(P^1)) &= \int_{-\infty}^{\infty} udP^2 - \int_{-\infty}^{\infty} udP^1 \\ &= \int_{-\infty}^{\infty} (P^1 - P^2)du \\ &\geq \int_{a}^{c} (P^1 - P^2)du \\ &\geq b[u(c) - u(a)] > 0 \end{aligned}$$

As for the axiom 3., we have:

$$\begin{aligned} u(CE[\alpha P^1 + (1-\alpha)P^3]) &= \alpha \mathbb{E}_{P^1}[u(P^1)] + (1-\alpha)\mathbb{E}_{P^3}[u(P^3)] \\ &= \alpha \mathbb{E}_{P^2}[u(P^2)] + (1-\alpha)\mathbb{E}_{P^3}[u(P^3)] \\ &= u(CE[\alpha P^1 + (1-\alpha)P^3) \end{aligned}$$

Hence all three axioms are satisfied.

Step 2 The main part of the proof:
Given lottery $P \in L$ and its range $[x_l, x_u]$, let's define a function:

$$v_{[x_l,x_u]}(t) = CE((1-t)E_{x_l} + tE_{x_u})$$

We will henceforth use a shorter notation $v(t)$, but bear in mind that this function is range-dependent. Then $v(0) = CE(E_{x_l}) = x_l$ and $v(1) = CE(E_{x_u}) = x_u$.
Let's assume provisionally that $v(t)$ is strictly increasing and continuous. Then $v(t)$ has an inverse:

$$u(x) = v^{-1}(x) = t$$

which is also continuous and increases strictly from 0 to 1 when x increases from x_l to x_u. Function $u(x)$ is also range dependent and we will later use full notation $u_{[x_l,x_u]}(x)$.

If

$$x = v(t), \quad t = u(x),$$

Then $CE(\beta_x) = x = v(t) = CE[(1-u(x))\beta_{x_l} + u(x)\beta_{x_u}]$. Using our two observations made prior to this proof, we obtain:

$$\begin{aligned}
u(CE(P)) &= u\left(CE\left[\sum_{i=1}^{n} p_i \beta_{x_i}\right]\right) \\
&= u\left(CE\left[\sum_{i=1}^{n} p_i \left\{(1-u(x_i))\beta_{x_l} + u(x_i)\beta_{x_u}\right\}\right]\right) \\
&= u\left(CE\left[\left(1 - \sum_{i=1}^{n} p_i u(x_i)\right)\beta_{x_l} + \left(\sum_{i=1}^{n} p_i u(x_i)\right)\beta_{x_u}\right]\right) \\
&= u\left(v\left(\sum_{i=1}^{n} p_i u(x_i)\right)\right) = u\left(u^{-1}\left(\sum_{i=1}^{n} p_i u(x_i)\right)\right) \\
&= \sum_{i=1}^{n} p_i u(x_i)
\end{aligned}$$

Step 3 It remains it be proved that function $v(t)$ is strictly increasing and continuous. We shall prove that it is strictly inreasing leaving the more technical part concerning continuity aside. The proof of this part may be found in Hardy et al. (1934) (page 162).

If $1 \leq t_1 < t_2 \leq 1$, then

$$(1-t_1)\beta_{x_l} + t_1\beta_{x_u} \geq (1-t_2)\beta_{x_l} + t_2\beta_{x_u}$$

for all x, with inequality for some x. Hence, by axiom 2.,

$$v(t_1) = CE((1-t_1)\beta_{x_l} + t_1\beta_{x_u}) < CE((1-t_2)\beta_{x_l} + t_2\beta_{x_u}) = v(t_2) \quad \square$$

Notice that for now we did not make use of the fourth axiom. Axioms 1.-3. are necessary and sufficient for range-dependent utility representation for a given simple lottery. To compare two simple lotteries we either need to construct two different range-dependent utilities (when the ranges differ) or to construct one range-dependent utility (when ranges coincide) and then calculate certainty equivalent for each of these lotteries on the basis of the relevant range-dependent utilities. Finally, the two certainty equivalents may be compared to rank the lotteries in order of preference. The only case left is when we are dealing with a compound lottery. If it is the case that simple lotteries being part of the bigger compound lottery have different ranges, we need to have a way to evaluate them. Axiom 4. states that in such case, a compound lottery is treated differently than the equivalent simple lottery. A compound lottery is evaluated in two steps: first simple lotteries are evaluated separately by using a possibly different range-dependent utilities and then we can reduce a compound lottery by substituting simple lotteries with their certainty equivalents. The resulting lottery will again be a simple lottery, and hence we can again apply the regular procedure for finding certainty equivalent for it by constructing yet another range-dependent utility with the new range being the interval between the smallest and the highest certainty equivalent from among the certainty equivalents for each simple lottery. This axiom has the implication that the compound lottery with simple lotteries inside having different ranges will be evaluated differently than the equivalent simple lottery.

Proof of theorem 2

This subsection contains the proof of the second main theorem concerning decision utility model (Theorem 2). Note first that x_l and x_u may be chosen arbitrarily and we will prefer to set $x_l = \min(\mathbf{x}), x_u = \max(\mathbf{x})$. We will write $x_{max} \equiv \max(\mathbf{x})$ and $x_{min} = \min(\mathbf{x})$ for simplicity. Also note that the theorem says that decision utility is the only way to get scale and shifts invariance without degenerating risk attitudes to the uninteresting case of risk neutrality which would be the case if we assumed Expected Utility theory.

Proof. ($\Longleftarrow$) part
Suppose that D is a range-dependent utility function (axioms 1-4). Then using the standard expression for the Ceratinty Equivalent such as in (9) we obtain that:

$$CE(\alpha\mathbf{x}+\beta) = D^{-1}\left[\mathbb{E}D(\alpha\mathbf{x}+\beta)\right]$$

On the other hand from the statement of the theorem we have that:

$$CE(\mathbf{x}) \quad = \quad x_{min} + (x_{max} - x_{min})D^{-1}\left[\mathbb{E}D\left(\frac{\mathbf{x} - x_{min}}{x_{max} - x_{min}}\right)\right]$$

Take

$$\alpha = \frac{1}{x_{max} - x_{min}}, \;\; \beta = -\frac{x_{min}}{x_{max} - x_{min}}$$

Hence after substituting it into $\alpha CE(\mathbf{x}) + \beta$ we get the desired conclusion that $CE(\alpha\mathbf{x}+\beta) = \alpha CE(\mathbf{x}) + \beta$ which means that the certainty equivalent is scale and shifts invariant.
($\Longrightarrow$) part

If utility function is range-independent as in Expected Utility Theory (universal shape for all outcomes), scale and shifts invariance implies that utility function is affine (risk neutrality). Let's consider range-dependent utilities. We already know that scale and shifts independence does not imply risk neutrality here (part $\Leftarrow$). We want to show that among range-dependent utilities, decision utility (universal shape applied for all ranges) is the only way to get scale and shifts invariance.

We will prove it by contradiction. Suppose that apart from decision utility there exists range-dependent utility which does not have a universal shape for all ranges and which has the property of scale and shifts invariance. Take $\mathbf{x}$ and $\mathbf{y} = \alpha\mathbf{x} + \beta$. Construct range-dependent utilities $u_{[x_{min},x_{max}]}$ and $u_{[y_{min},y_{max}]}$. Notice that to obtain the certainty equivalent of lottery $\mathbf{y}$ we can either calculate it directly via:

$$CE(\mathbf{y}) = u^{-1}_{[y_{min},y_{max}]}\left[\mathbb{E}u_{[y_{min},y_{max}]}(\mathbf{y})\right]$$

Or alternatively do it indirectly via an auxiliary utility function $u^*_{[x_{min},x_{max}]}(\cdot)$, which is defined in the following way:

$$u^*_{[x_{min},x_{max}]}(x) = u_{[y_{min},y_{max}]}(y)$$

where $y = \alpha x + \beta$ are respective lottery outcomes.

After defining function $u^*_{[x_{min},x_{max}]}(\cdot)$, we should then calculate the normalized certainty equivalent r:

$$r = u^{*-1}_{[x_{min},x_{max}]}\left[\mathbb{E}u^*_{[x_{min},x_{max}]}(\mathbf{x})\right]$$

And finally we should denormalize r back to the original range of values by:

$$CE(\mathbf{y}) = \alpha r + \beta$$

Or alternatively:

$$CE(\mathbf{y}) = \beta + \alpha u^{*-1}_{[x_{min},x_{max}]}\left[\mathbb{E}u^*_{[x_{min},x_{max}]}\left(\frac{\mathbf{y}-\beta}{\alpha}\right)\right]$$

Since our range dependent utilities are not universal for each shape by assumption, we require that functions $u^*_{[x_{min},x_{max}]}(\cdot)$ and $u_{[x_{min},x_{max}]}$ differ in shape for at least one argument:

$$\exists x' \in [x_{min}, x_{max}] : \quad u^*_{[x_{min},x_{max}]}(x') \neq u_{[x_{min},x_{max}]}(x')$$

As a result it is always possible to find such lottery $\mathbf{x}$, for which:

$$r \neq CE(\mathbf{x})$$

Hence $CE(\mathbf{y}) \neq \alpha CE(\mathbf{x}) + \beta$ or:

$$CE(\alpha\mathbf{x} + \beta) \neq \alpha CE(\mathbf{x}) + \beta$$

□

Proof of proposition 5.1

We now prove proposition 5.1.

Proof. Let's take the following 3-outcome lottery $L \equiv (x_1, p_1; x_2, p_2; x_3, p_3)$. Analysing 3-outcome lottery turns out to be sufficiently general to obtain monotonicity result which is the aim of this section. Similarly as in the main text, let's define η, the relative position of a "x_2" outcome and r, the relative certainty equivalent as:

$$\eta = \frac{x_2 - x_1}{x_3 - x_1}, \quad r: \; D(r) = p_3 + p_2 D(\eta)$$

The relation between $D(r)$ and $D(\eta)$ may be presented on the following graph: Now we obtain the formula for certainty equivalent of L within the decision

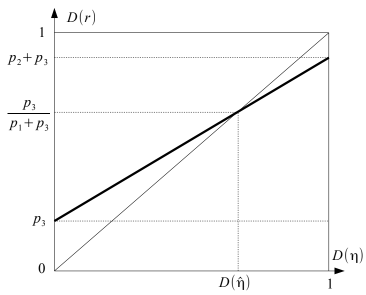

Figure 9: Linear relation between $D(r)$ and $D(\eta)$

utility model:

$$CE(L) = x_1 + (x_3 - x_1)r] \tag{22}$$

To find conditions under which decision utility does not exhibit monotonicity violations we need to analyze the following two derivatives:

$$\frac{dCE(L)}{dx_1} = (1 - r) - p_2 \frac{(1 - \eta)D'(\eta)}{D'(r)}$$

$$\frac{dCE(L)}{dx_3} = r - p_2 \frac{\eta D'(\eta)}{D'(r)}$$

In order to exclude monotonicity violations we demand that the above derivatives be, respectively, non-negative and non-positive. Equivalently we demand that the following two functions be non-negative:

$$f(\eta) \quad = \quad rD'(r) - p_2\eta D'(\eta) \geq 0 \tag{23}$$

$$g(\eta) \quad = \quad (1-r)D'(r) - p_2(1-\eta)D'(\eta) \geq 0 \tag{24}$$

To analyze the properties of the above to functions we take further derivative with respect to η:

$$\frac{df}{d\eta} = p_2 D'(\eta)\left[-\frac{\eta D''(\eta)}{D'(\eta)} - \left(-\frac{rD''(r)}{D'(r)}\right)\right]$$

$$\frac{dg}{d\eta} = p_2 D'(\eta)\left[\frac{(1-r)D''(r)}{D'(r)} - \frac{(1-\eta)D''(\eta)}{D'(\eta)}\right]$$

Using the definitions of Relative Risk Aversion and Inverse Relative Risk Aversion we can now rewrite the above derivatives as follows:

$$\frac{df}{d\eta} = p_2 D'(\eta)([RRA(\eta) - RRA(r)]$$

$$\frac{dg}{d\eta} = p_2 D'(\eta)\left[InvRRA(r) - InvRRA(\eta)\right]$$

Notice that there exists value η for which $\eta = \hat{\eta} = r$ (See figure 7), so that the above two derivatives are zero. It means that this point is a candidate for a local extremum for both functions f and g. Let's calculate second derivatives:

$$\frac{d^2 f}{d\eta^2} = p_2 D''(\eta)\left[RRA(\eta) - RRA(r)\right] + p_2 D'(\eta)\left[RRA(\eta) - RRA(r)\frac{p_2 D'(\eta)}{D'(r)}\right]$$

$$\frac{d^2 g}{d\eta^2} = p_2 D''(\eta)\left[InvRRA(r) - InvRRA(\eta)\right] + p_2 D'(\eta)\left[InvRRA(r)\frac{p_2 D'(\eta)}{D'(r)} - InvRRA(\eta)\right]$$

For $\eta = \hat{\eta}$ we obtain the following formulae:

$$\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=\hat{\eta}} = 0 + p_2 D'(\hat{\eta})RRA'(\hat{\eta})(p_1 + p_3)$$

$$\left.\frac{d^2 g}{d\eta^2}\right|_{\eta=\hat{\eta}} = 0 - p_2 D'(\hat{\eta})InvRRA'(\hat{\eta})(p_1 + p_3)$$

Hence we obtain the following conditions:

$$\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=\hat{\eta}} \gtreqless 0 \quad \text{if} \quad RRA'(\hat{\eta}) \gtreqless 0$$

$$\left.\frac{d^2 g}{d\eta^2}\right|_{\eta=\hat{\eta}} \gtreqless 0 \quad \text{if} \quad InvRRA'(\hat{\eta}) \lesseqgtr 0$$

Suppose that $RRA'(\eta) \geq 0$ and $InvRRA'(\eta) \leq 0$ for all η. Then $\eta = \hat{\eta}$ is a global minimum (not necessarily unique) of both functions f and g. In this case monotonicity property is satisfied because:

$$\begin{aligned} f(\eta) \geq f(\hat{\eta}) &= (p_1 + p_3)\hat{\eta}D'(\hat{\eta}) > 0 \\ g(\eta) \geq g(\hat{\eta}) &= (p_1 + p_3)(1 - \hat{\eta})D'(\hat{\eta}) > 0 \end{aligned}$$

Suppose now that $RRA'(\eta) < 0$ and $InvRRA'(\eta) > 0$ for all η. Then $\eta = \hat{\eta}$ is a global maximum (not necessarily unique) of both functions f and g. In this case, we have to check whether f and g assume positive values on the borders of its domains, i.e. when $\eta \to 1$ and $\eta \to 0$. The respective function values are as follows:

$$\begin{aligned} \lim_{\eta\to 0} f(\eta) &= D^{-1}(p_1)D'[D^{-1}(p_1)] - p_2 \lim_{\eta\to 0} \eta D'(\eta) \\ \lim_{\eta\to 0} g(\eta) &= [1 - D^{-1}(p_1)]D'[D^{-1}(p_1)] - p_2 \lim_{\eta\to 0}(1-\eta)D'(\eta) \\ \lim_{\eta\to 1} f(\eta) &= D^{-1}(1-p_3)D'[D^{-1}(1-p_3)] - p_2 \lim_{\eta\to 1} \eta D'(\eta) \\ \lim_{\eta\to 0} g(\eta) &= [1 - D^{-1}(1-p_3)]D'[D^{-1}(1-p_3)] - p_2 \lim_{\eta\to 1}(1-\eta)D'(\eta) \end{aligned} \tag{25}$$

Suppose now that p_1 and p_3 tend to 0, i.e. when it is most difficult to make the above expressions nonnegative. Then the sufficient conditions that these function values are nonnegative are:

$$\begin{aligned} \lim_{\eta\to 0}[\eta D'(\eta)]' \geq 0 &\iff \lim_{\eta\to 0} RRA(\eta) \leq 1 \\ \lim_{\eta\to 0}[(1-\eta)D'(\eta)]' \geq 0 &\iff \lim_{\eta\to 0} InvRRA(\eta) \geq 1 \\ \lim_{\eta\to 1}[\eta D'(\eta)]' \leq 0 &\iff \lim_{\eta\to 1} RRA(\eta) \geq 1 \\ \lim_{\eta\to 1}[(1-\eta)D'(\eta)]' \leq 0 &\iff \lim_{\eta\to 1} InvRRA(\eta) \leq 1 \end{aligned}$$

It is now apparent that it is not possible to satisfy the above conditions and at the same time to be the case that: $RRA'(\eta) < 0$ and $InvRRA'(\eta) > 0$ for all η. Hence we can exclude the possibility that f and g have a global maximum in $\eta = \hat{\eta}$. □

Appendix 2

Table 2: Median (for 25 subjects) Certainty Equivalents (CE) for 28 **prospects involving GAINS**, taken from Tversky and Kahneman (1992). Each prospect offers xu with probability p and xl with the remaining probability. For example median Certainty Equivalent for lottery ($\$\mathbf{0}, \mathbf{0.9}, \$\mathbf{50}, \mathbf{0.1}$) is equal to $\$\mathbf{9}$. Additionally, predicted theoretical values for CE together with the Sum of Squared Errors (the bottom raw) using different estimated models (all models were fitted with help of the cumulative beta distribution function) are presented: **EV** - Expected Value of a given prospect, **EU** - Expected Utility: one universal fitted utility function in the whole range $[0, 400]$, **DU1** - All Certainty Equivalents are first normalized to the range $[0, 1]$ and then used to estimate the decision utility function. The resulting decision utility function is then applied in the model (13), **DU2** - The model (13) is directly estimated, **RDU** - Range-dependent utility - for each range a different utility function is fitted, **PT** - Prospect Theory - the original parameters estimated by Tversky and Kahneman (1992) are used. The last two columns present an instance-wise comparison of decision utility model (DU1) with Expected Utility and with Prospect Theory. A cross denotes an instance in which DU1 gives better prediction than the counterpart model.

No	xu	xl	p	CE	EV	EU	DU1	DU2	RDU	PT	DU1-EU	DU1-PT
1	50	0	0.10	9.0	5.0	1.8	7.6	6.0	8.5	7.4	x	x
2	50	0	0.50	21.0	25.0	18.0	20.3	19.7	21.7	18.7	x	x
3	50	0	0.90	37.0	45.0	42.7	35.1	36.5	36.6	34.0	x	x
4	100	0	0.05	14.0	5.0	1.2	10.5	7.5	10.3	10.0	x	x
5	100	0	0.25	25.0	25.0	12.6	25.8	23.0	24.8	24.6	x	
6	100	0	0.50	36.0	50.0	34.7	40.6	39.4	38.8	37.4		
7	100	0	0.75	52.0	75.0	63.8	56.6	57.8	54.2	52.6	x	
8	100	0	0.95	78.0	95.0	92.2	77.2	80.5	74.6	76.9	x	x
9	200	0	0.01	10.0	2.0	0.2	9.2	5.3	4.5	7.4	x	x
10	200	0	0.10	20.0	20.0	6.0	30.6	24.1	21.4	29.6	x	
11	200	0	0.50	76.0	100.0	63.8	81.1	78.8	73.2	74.8	x	
12	200	0	0.90	131.0	180.0	164.9	140.3	145.9	140.1	135.9	x	
13	200	0	0.99	188.0	198.0	196.2	175.0	181.4	177.8	180.0		
14	400	0	0.01	12.0	4.0	0.3	18.3	10.5	12.0	14.9	x	
15	400	0	0.99	377.0	396.0	369.2	350.0	362.7	377.0	360.1		
16	100	50	0.10	59.0	55.0	54.3	57.6	56.0	59.3	59.0	x	
17	100	50	0.50	71.0	75.0	73.1	70.3	69.7	70.5	70.5	x	
18	100	50	0.90	83.0	95.0	94.3	85.1	86.5	83.2	85.2	x	x
19	150	50	0.05	64.0	55.0	53.8	60.5	57.5	59.3	62.4	x	
20	150	50	0.25	72.5	75.0	70.3	75.8	73.0	73.7	77.7		x
21	150	50	0.50	86.0	100.0	93.4	90.6	89.4	88.0	90.5	x	
22	150	50	0.75	102.0	125.0	119.8	106.6	107.8	103.9	105.3	x	
23	150	50	0.95	128.0	145.0	143.6	127.2	130.5	124.9	128.4	x	
24	200	100	0.05	118.0	105.0	104.0	110.5	107.5	115.1	112.7	x	
25	200	100	0.25	130.0	125.0	121.0	125.8	123.0	131.0	128.2	x	
26	200	100	0.50	141.0	150.0	144.4	140.6	139.4	144.9	141.1	x	
27	200	100	0.75	162.0	175.0	170.4	156.6	157.8	159.3	155.8	x	x
28	200	100	0.95	178.0	195.0	193.8	177.2	180.5	177.6	178.7	x	
				SSE	6767.0	4096.9	1402.0	929.2	338.4	656.6	24	9

Table 3: Median (for 25 subjects) Certainty Equivalents (CE) for 28 **prospects involving LOSSES**, taken from Tversky and Kahneman (1992). Each prospect offers xu with probability p and xl with the remaining probability. For example median Certainty Equivalent for lottery ($\mathbf{\$0, 0.9, -\$50, 0.1}$) is equal to $-\$\mathbf{8}$. Additionally, predicted theoretical values for CE together with the Sum of Squared Errors (the bottom raw) using different estimated models (all models were fitted with help of the cumulative beta distribution function) are presented: **EV** - Expected Value of a given prospect, **EU** - Expected Utility: one universal fitted utility function in the whole range $[0, 400]$, **DU1** - All Certainty Equivalents are first normalized to the range $[0, 1]$ and then used to estimate the decision utility function. The resulting decision utility function is then applied in the model (13), **DU2** - The model (13) is directly estimated, **RDU** - Range-dependent utility - for each range a different utility function is fitted, **PT** - Prospect Theory - the original parameters estimated by Tversky and Kahneman (1992) are used. The last two columns present a instance-wise comparison of decision utility model (DU1) with Expected Utility and with Prospect Theory. A cross denotes an instance in which DU1 gives better prediction than the counterpart model.

No	xu	xl	p	CE	EV	EU	DU1	DU2	RDU	PT	DU1-EU	DU1-PT
1	50	0	0.10	8.0	5.0	3.8	6.7	6.2	7.2	6.7	x	
2	50	0	0.50	21.0	25.0	22.7	21.1	21.3	21.9	20.4	x	x
3	50	0	0.90	39.0	45.0	44.3	37.8	38.9	38.4	37.4	x	x
4	100	0	0.05	8.0	5.0	3.4	8.5	7.6	7.4	8.3	x	
5	100	0	0.25	23.5	25.0	20.2	25.0	24.5	24.1	24.8	x	
6	100	0	0.50	42.0	50.0	44.2	42.1	42.7	42.3	40.8	x	x
7	100	0	0.75	63.0	75.0	70.9	60.7	62.4	62.0	58.8	x	x
8	100	0	0.95	84.0	95.0	94.0	82.8	84.9	84.7	83.1	x	
9	200	0	0.01	3.0	2.0	1.0	6.1	4.9	4.7	5.1		
10	200	0	0.10	23.0	20.0	13.2	26.7	24.9	24.5	26.7	x	x
11	200	0	0.50	89.0	100.0	82.6	84.2	85.3	86.3	81.5	x	x
12	200	0	0.90	155.0	180.0	172.8	151.2	155.5	157.6	149.7	x	x
13	200	0	0.99	190.0	198.0	197.2	184.3	187.5	188.8	187.6	x	
14	400	0	0.01	14.0	4.0	1.5	12.1	9.9	14.0	10.2	x	x
15	400	0	0.99	380.0	396.0	374.9	368.7	375.0	380.0	375.1		
16	100	50	0.10	59.0	55.0	54.6	56.7	56.2	58.9	58.2	x	
17	100	50	0.50	71.0	75.0	74.0	71.1	71.3	71.1	72.2	x	x
18	100	50	0.90	85.0	95.0	94.6	87.8	88.9	84.9	88.4	x	x
19	150	50	0.05	60.0	55.0	54.3	58.5	57.6	57.6	60.4	x	
20	150	50	0.25	71.0	75.0	72.2	75.0	74.5	74.0	78.0		x
21	150	50	0.50	92.0	100.0	96.2	92.1	92.7	91.6	93.8	x	x
22	150	50	0.75	113.0	125.0	122.0	110.7	112.4	110.9	111.2	x	
23	150	50	0.95	132.0	145.0	144.2	132.8	134.9	133.5	134.2	x	x
24	200	100	0.05	112.0	105.0	104.3	108.5	107.6	109.1	110.7	x	
25	200	100	0.25	121.0	125.0	122.3	125.0	124.5	124.7	128.5		x
26	200	100	0.50	142.0	150.0	146.1	142.1	142.7	140.4	144.4	x	x
27	200	100	0.75	158.0	175.0	171.9	160.7	162.4	157.7	161.7	x	x
28	200	100	0.95	179.0	195.0	194.2	182.8	184.9	179.2	184.5	x	x
				SSE	2801.3	1841.0	323.2	211.0	70.6	353.8	24	17