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# Lottery valuation using the aspiration / relative utility function

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**Abstract**

The paper presents a method for lottery valuation using the relative utility function. This function was presented by Kontek (2009) as "the aspiration function" and resembles the utility curve proposed by Markowitz (1952A). The paper discusses lotteries with discrete and continuous outcome distributions as well as lotteries with positive, negative and mixed outcomes providing analytical formulas for certainty equivalents in each case. The solution is similar to the Expected Utility Theory approach and does not use the probability weighting function – one of the key elements of Prospect Theory. Solutions to several classical behavioral problems, including the Allais paradox, are presented, demonstrating that the method can be used for valuing lotteries even in more complex cases of outcomes described by a combination of Beta distributions. The paper provides strong arguments against Prospect Theory as a model for describing human behavior and lays the foundations for Relative Utility Theory – a new theory of decision making under conditions of risk.

**JEL classification**: D03, D81

**Keywords:** Lottery Valuation, Expected Utility Theory, Markowitz Hypothesis, Prospect / Cumulative Prospect Theory, Aspiration / Relative Utility Function.

## 1. Introduction.

**1.1.** Lottery valuation is one of the fundamental problems of examining decision making under conditions of risk and uncertainty. The earliest method, based on the expected value of the lottery, was replaced during the first half of the 18th century by Bernoulli (1738) and Cramer (1728), who adopted the concept of outcome utility. This approach was vindicated by the Expected Utility (EU) Theory of von Neumann and Morgenstern (1944), who derived a method of lottery valuation from a few axiomatic assumptions regarding the rationality of the decisions people make. An increasing number of psychological experiments since then, however, have pointed to the irrationality of human decision making, where "irrationality" was understood to mean any deviation from the axioms underlying EU Theory. The concept of probability weighting, allowing to explain certain experimental results, led to the development of Prospect Theory (Kahneman, Tversky, 1979) and its Cumulative version (Tversky, Kahneman, 1992). This resulted in even more elaborate methods of calculating the hypothetical lottery "value".

However, Kontek (2009) demonstrated that the same experimental data used to derive Cumulative Prospect Theory (CPT) can lead to a completely different solution, one which resembles the utility function hypothesized by Markowitz (1952A). This meant that the concept of probability weighting was no longer needed to explain the experimental results. Kontek named the resulting curve "the aspiration function", because human aspirations well explain risk aversion and risk seeking during decision making. The resulting curve may be used to value lotteries, which is the main topic of the present paper. However, the term "aspiration function" is used interchangeably with "relative utility function" in this paper. The latter is used even more often as it is more apposite in the context of the historical discussion on valuing lotteries.

**1.2.** The present paper discusses lotteries with discrete and continuous outcome distributions and with positive, negative and mixed outcomes. Certainty equivalents are determined using the relative utility function and can serve to compare lotteries without establishing their hypothetical "values". Despite the appellation "relative utility function" the concept of utility is not actually introduced here as the function is expressed in terms of probability.[1] The entire discourse on lottery valuations is thus reduced to a discourse on the basis of classical probability theory. Moreover, the obtained solutions resemble the valuation

[1] Of fulfilling aspirations, achieving full satisfaction, reaching full success, or assuring the maximum subjective value of any event etc.

method of Expected Utility Theory and do not use the probability weighting function – a key element of Prospect Theory. The paper discusses several classical problems, including the Allais paradox and symmetrical bets, and presents solutions thereto using the relative utility function. It is shown that this method allows certainty equivalents to be analytically determined, even in more complex cases of outcomes with continuous distribution.

**1.3.** This paper is organized as follows. Point 2 defines some basic concepts and presents historical valuation theories. Point 3 outlines Prospect Theory and its Cumulative version. Point 4 discusses the aspiration / relative utility function. Point 5 derives the method for calculating certainty equivalents for discrete outcomes. Point 6 is completely devoted to analyzing the Allais paradox and presents a solution using the relative utility function. Point 7 discusses the determination of certainty equivalents for negative and mixed lotteries. Point 8 is dedicated to analyzing symmetrical bets. It is shown that the solution obtained by using the relative utility function predicts the experimental results much better than does the solution provided by CPT. Point 9 discusses the calculation of certainty equivalents in the case of continuous outcome distribution and presents several examples of more complex cases. Point 10 extends the results to mixed lotteries. Point 11 lays the foundations for Relative Utility Theory – a new theory of decision making under conditions of risk. Point 12 summarizes the results of the paper.

## 2. Basic Definitions and Expected Utility Theorem.

**2.1.** Let us assume that a set of outcomes X partitions the possible states of the world into mutually exclusive events. Let us then define a vector of lottery outcomes $x = \{x_1, x_2, \dots, x_n\}$ such that $x_1, x_2, \dots, x_n \in X$, and a vector of associated lottery probabilities $p = \{p_1, p_2, \dots, p_n\}$ such that $\sum_{i=1}^{n} p_i = 1$, where $p_i$ is the probability of outcome $x_i$. A lottery $y$ is therefore defined as a pair of vectors $y = \{x, p\}$. Additionally, let us define a set $Y$ of lotteries $y_i$, where the number of outcomes $n$ is in general different for each lottery. Using the above notation a certainty equivalent can be represented as $\{\{CE\}, \{1\}\}$. The expected value of a lottery can be determined similarly to any random event:

$$EX = \sum_{i=1}^{n} x_i p_i = x.p \tag{1}$$

Function (1) expresses the sum of the components as the scalar product of the outcome and probability vectors, which is a handy form for both notation and performing calculations.

**2.2** The expected value of a lottery with a continuous outcome distribution is determined by the formula:

$$EX = \int_{-\infty}^{\infty} xP(x)dx \qquad (2)$$

where $P(x)$ is the probability density function of the lottery outcomes. The first part of this paper discusses discrete outcomes. The results are later generalized to cover continuous distributions.

**2.3.** One of the problems encountered when calculating expected values is that an infinite sum does not converge for every series. The St. Petersburg paradox is a case in point. The solution put forward by Bernoulli in 1738 used logarithms of the outcomes instead of the outcomes themselves. The logarithms of the outcomes were an estimate of their utility to a human being and enabled a lottery value to be finite[2]. A similar solution was obtained 10 years earlier by Cramer, only he adopted a quadratic utility function. Therefore, assuming there exists a certain utility function $U$, let us determine an outcome utility vector $u = \{u_1, u_2, \ldots, u_n\}$, where $u_i = U(x)$, which enables calculation of the lottery value $V$ as:

$$V(y) = \sum_{i=1}^{n} U(x_i)p_i = \sum_{i=1}^{n} u_i p_i = u.p \qquad (3)$$

Neither Bernoulli nor Cramer ever considered a question of rationality when presenting their solutions. However this method of valuing lotteries was later vindicated by von Neumann and Morgenstern (1944) when their Expected Utility Theory derived identical solution from axioms which assumed that people made rational decisions.

**2.4.** The foregoing arguments against expected value and in favor of the expected utility approach are frequently encountered in the literature on the subject. However, the requirement that the best choice is sought by using the expected utility form (3) can be presented from a different angle. Markowitz (1959B) provided a convincing argument when analyzing portfolio selection: *An investor who sought only to maximize the expected return would never prefer a diversified portfolio. If one security had greater expected return than any other, the investor would place all his funds in this security. ... Thus if we consider diversification a sound principle of investment, we must reject the objective of simply maximizing the expected return*. It is undeniably true that outcome distribution is an important consideration for any decision maker. People most often try to avoid volatility but it also

[2] Of course it does not happen in all cases.

happens they seek it out. This means that the expected value of outcomes is an insufficient basis for rational decision making (however "rational" may be defined here).

**2.5.** The rational decision maker therefore has two options available. The first is to compile a list of criteria for consideration, expected value and variance being possible examples. This may arise from the simple assumption that the outcome distribution is normal and can therefore be described by these two parameters. However, outcome distribution need not be normal in the general case. In order to avoid this pitfall, the rational decision maker could include additional distribution criteria (such as the third and fourth distribution moments of skewness and kurtosis) or adopt other measures, such as Value at Risk (VaR) to help find the best choice. This shows an alternative way of lottery valuation and/or setting up preferences between them, without using the utility concept.

However, at a certain point while drawing up criteria lists, the rational decision maker will hit upon another way of proceeding. The obvious conclusion would be that the most general approach is to assign a value reflecting his or her *degree* of preference to each outcome. These values would plot a curve traditionally known as a utility function. A utility function is thus the most general way of describing people's preferences regarding lottery outcomes. It follows that applying Equation (3) should thus lead to the most rational decision, provided that the utility function correctly describes the rationality of the decision maker.

**2.6.** The utility function may be the most general solution but this does not necessarily mean that it is the most convenient to use. However, knowing it or merely assuming its shape may lead to a much simpler way of proceeding. Markowitz (1952B, 1959) proves that if a utility function can be approximated by a quadratic function, then the maximum expected utility for any outcome distribution can be found out by only taking the expected value and the variance into consideration. This is the assumption underlying the Markowitz Portfolio Theory (1952B, 1959) which coincides with considerations of the same mean and variance resulting from the assumption of normal outcome distribution, basically without knowledge of the utility function.

## 3. Prospect / Cumulative Prospect Theory.

**3.1.** Despite several theoretical arguments presented in section 2 which support the Expected Utility approach, many experiment results pointed out the irrationality of human behavior. It is important to note that irrationality was here understood to mean any discrepancy between the decisions that were made and the axioms of EU. One example was the Allais paradox (1952), which did not satisfy the independence axiom. This deviation could be explained by people having a nonlinear perception of probabilities, which led to the concept of probability weighting presented by Edwards (1961). Handa (1977) proposed a solution symmetrical to Function (3):

$$V(y) = \sum_{i=1}^{n} x_i W(p_i) = \sum_{i=1}^{n} x_i w_i = x.w \qquad (4)$$

where $W(p)$ is the probability weighting function such that $W(0) = 0$, $w_i = W(p_i)$ are the weights assigned to specific probabilities, and $w$ is a vector of weighted probabilities $w = \{w_1, w_2, \ldots, w_n\}$.

**3.2.** Kahneman and Tversky (1979) combined the approaches described by Functions (3) and (4) supra in their Prospect Theory to derive:

$$V(y) = \sum_{i=1}^{n} U(x_i) W(p_i) = \sum_{i=1}^{n} u_i w_i = u.w \qquad (5)$$

Incorporating the concept of probability weighting into Prospect Theory is subject to a major caveat. This idea, which helped explain the Allais paradox, and some of the experiments conducted by Kahneman and Tversky, coincided with theories stating that people perceive probabilities in a subjective way (Subjective Expected Utility Theory, Savage, 1954). This led to the commonly accepted view that decision weights and the probability weighting function have a profound psychological justification. This is, however, not the case. Subjective probabilities are mathematically indistinguishable from classical definitions of probability and express degrees of belief in situations when exact probabilities are *unknown*. In the Allais paradox (and the other experiments discussed), per contra, the probabilities are *known* exactly. As people have never been observed to perceive *known* probabilities in a distorted way, Prospect Theory states: "*decision weights are not probabilities, they do not obey the probability axioms and they should not be interpreted as measures of degree or belief*" (Kahneman, Tversky, 1979). Prospect Theory thus assumes that people distort probabilities (even known ones) when making decisions without, however, offering any psychological explanation as to how this is possible. Nor has any mechanism ever been posited to explain why this probability transformation effect only manifests itself at the moment a decision is made.

Prospect Theory thus describes (rather than explains) the experimental results by means of an artificial concept whose basis is more mathematical than psychological[3]. It is interesting to note that Kahneman and Tversky clearly distinguished *overestimation* (often

[3] This can be seen even more clearly in the case of Cumulative Prospect Theory whose axiomatization is based on complex topological models using Choquet integrals (Refer point 3.4. and 3.5. for CPT).

encountered when assessing the probability of rare events) from *overweighting*, the latter being a mere *feature* of decision weights. Despite this, *overweighting* is commonly treated as a "psychological" explanation of people's behavior.

**3.3.** The concept of probability weighting leads anyhow to another serious problem concerning stochastic dominance. This problem boils down to the paradox that an outcome of $x_i$, having a probability of $p_i$, and the same outcome, when broken into $k$ components, each having a probability of $p_i / k$ give different contributions to the calculated lottery value. This makes the value of a lottery dependent on how it is represented. As an example, let us take an outcome of $100 with a probability of 0.5 broken down into 5 outcomes of $100 with probability 0.1 each. Assuming the overweighting of small probabilities postulated by Prospect Theory, the five outcomes calculated separately give a greater lottery value than when these outcomes are treated as a single lottery component. To put it another way, a lottery represented by 5 components would dominate over the same lottery represented by one component, which is a nonsensical result.

Prospect Theory assumes that problems like this can be avoided with the assistance of mental operations carried out during the editing phase of decision making, i.e. before the proper lottery valuation. More specifically, the *Combination* operations merge the components of the same outcome into a single component with a combined probability. Additionally, as a result of *Simplification* operations, values of outcomes and probabilities are rounded, which in case of their similarity can lead to further combinations of lottery components. Finally, in order to preempt further dispute and dispel all doubt concerning division of components, Prospect Theory assumes that the weighting function is discontinuous for probabilities approaching 0 and 1. This means that it is not possible to determine weights for events which are almost impossible or almost certain. This method of "editing out" the potential difficulties, however, did not eliminate other problems, such as the sum of probability weights not equaling 1. This means that they cannot be treated as probabilistic measures, which obviously complicates the axiomatization of the theory (Wakker, 1989, Tversky, Kahneman 1992).

**3.4.** A solution to these problems was proposed by Quiggin in 1982. Quiggin, in his Rank-Dependent Expected Utility Theory (which he called Anticipated Utility Theory) assumed that the probability weighting function does not relate to individual probabilities, but to their cumulative values. This approach can be presented as follows. A probability vector $p = \{p_1, p_2, \ldots, p_n\}$ is ordered by the ranking of outcomes $x_i$ (e.g. $x_1 > x_2 > \ldots > x_n$). A cumulative probability vector $cp = \{cp_1, cp_2, cp_3, \ldots, cp_n\} = \{p_1, p_1 + p_2, p_1 + p_2 + p_3, \ldots, 1\}$

is then introduced. Next, by applying the probability weighting function $W$, a weighted cumulative probability vector $w = \{w_1, w_2, …., 1\}$ is obtained where $w_i = W(cp_i)$. Finally, a vector of weighted cumulative probability increments is defined such that $\Delta w = \{\Delta w_1, \Delta w_2, …, \Delta w_n\} = \{w_1, w_2 - w_1, w_3 - w_2, …, 1 - w_{n-1}\}$. This is now used to calculate the lottery value:

$$V(y)=\sum_{i=1}^{n}U(x_i)\Delta\left(W\left(\sum_{k=1}^{i}p_k\right)\right)=\sum_{i=1}^{n}U(x_i)\Delta(W(cp_i))=\sum_{i=1}^{n}U(x_i)\Delta(w_i)=\sum_{i=1}^{n}u_i\Delta w_i=u.\Delta w \tag{6}$$

It is obvious that the lottery value obtained using the proposed methodology depends on the order of the components in the $x$ and $p$ vectors. This is because the cumulative probability vector and any subsequent results depend on how the outcomes are ranked. Some serious objections may be raised, even leaving aside the complexity of the methodology. Why do the $x$ and $p$ vectors have to be ordered at all? And, if they do have to be ordered, why choose any particular order over any other?

**3.5.** Despite not having given a convincing answer to that question, the procedure proposed by Quiggin was adopted virtually unchanged by Cumulative Prospect Theory introduced by Tversky and Kahneman in 1992. It should be noted that it is not necessary to go through the cumulative probabilities phase for lotteries with two outcomes, according to CPT. This is a corollary of the following relationships: if $p = \{p_1, 1 - p_1\}$ then $cp = \{p_1, 1\}$, and therefore $w = \{w_1, 1\}$ and $\Delta w = \{w_1, 1 - w_1\}$. It turns out that the weight of the main outcome probability $w_1 = W(p_1)$ is all that is required to calculate the lottery value. This is even more visible in the case of a lottery with a single prize (with the second outcome of 0). Equation (6) then simplifies to:

$$V(y)=u.\Delta w=u_1w_1 \tag{7}$$

This means that CPT simplifies to the original theory from 1979 for a lottery with two outcomes. Apart from introducing the methodology described above, Cumulative Prospect Theory, on the basis of experimental data, determined the shape of the utility function $U(x)$ (which the theory calls the value function):

$$U(x)=\lambda|x|^{\alpha} \tag{8}$$

and the probability weighting function $W(p)$:

$$W(p)=\frac{p^{\gamma}}{\left(p^{\gamma}+(1-p)\gamma\right)^{1/\gamma}} \tag{9}$$

where $\lambda = 1$ and $\gamma = 0.61$ for positive prospects, and $\lambda = -2.25$ i $\delta = 0.69$ for negative prospects ($\delta$ replaces $\gamma$ in Formula (9) used for losses), whereas $\alpha = 0.88$ is valid for both positive and negative prospects.

**3.6.** Cumulative Prospect Theory compares lotteries by using their values *V*, as described by Function (6). Such a comparison, however, can be also made on the basis of their certainty equivalents, although this would require however further calculations. Transforming in turn yields:

$$V(y) = V(CE) = U(CE)W(1) = U(CE) \tag{10}$$

and applying Function (8) gives:

$$CE = \left(\frac{V(y)}{\lambda}\right)^{1/\alpha} \tag{11}$$

where the lottery value $V(y)$ is taken from Function (6).

## 4. Aspiration / Relative Utility Function.

**4.1.** By analyzing the same data used to derive Cumulative Prospect Theory, Kontek (2009) came up with a completely different solution (Fig.1, left), one whose shape resembled the utility function proposed by Markowitz in his 1952 paper "The Utility of Wealth" (Fig. 1, right). The obtained curve, which Kontek called the aspiration function, explains the results of the experiments conducted by Kahneman and Tversky without using a probability weighting function. The function is expressed in terms of probability (of fulfilling aspirations, achieving full satisfaction, reaching full success, or assuring the maximum subjective value of any event etc.). This way the concept of aspiration offers an elegant explanation of risk-seeking and risk-aversion attitudes when making decisions under conditions of risk. The terms "aspiration function" and "relative utility function" are used interchangeably in this paper, the latter being justified by virtue of its historical association with lottery valuation research.

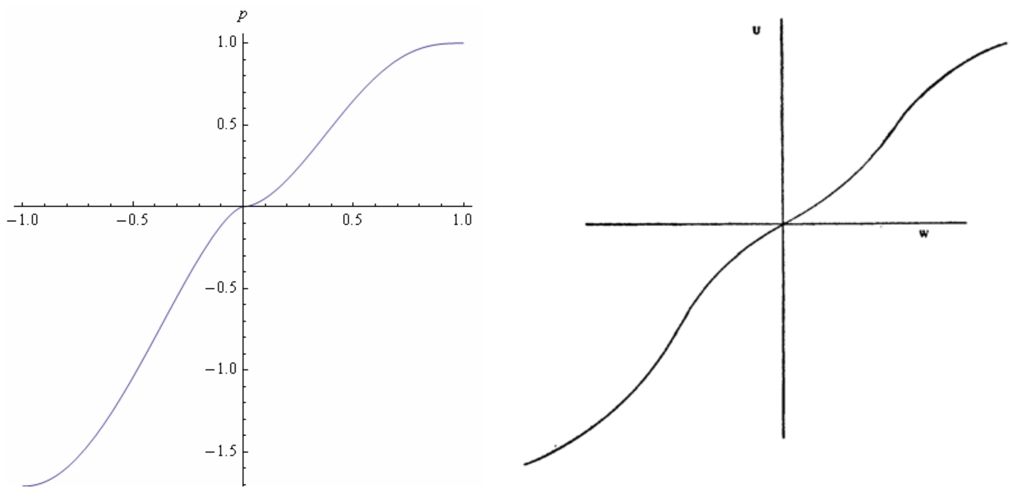


Fig. 1. Aspiration / relative utility function $p = Q(r)$ (Left); Utility function as presented in the Markowitz hypothesis (1952A) (Right).

Depending on the $x$ scale, the relative utility function can be determined as $p = Q(r)$, where $r$ is the relative certainty equivalent or $p = U(s)$, where $s$ is the normalized logarithm of $r$ [4]. Considering the relative certainty equivalent (instead of its absolute value) stems from the process of focusing attention (Kontek, 2009). The transposition (shift) of $x$ by the risk free component $A$, defined as the outcome with the lowest absolute value[5], has to be taken into account when determining the value of variable $r$. This shift results from the process of mental adaptation (Kontek, 2009) and from operations carried out during the Editing phase, which concept was introduced by Prospect Theory. Thus:

$$r = \frac{CE - A}{P - A} \tag{12}$$

where $CE$ is the certainty equivalent and $P$ is the maximum outcome of the lottery. A corollary of the above relationship is that $r$ assumes values from the range [0, 1], even in the case of lotteries with a risk free component. For $A = 0$, Function (12) reduces to:

$$r = \frac{CE}{P} \tag{13}$$

Solving (12), the certainty equivalent value $CE$ of a lottery can be determined as:

$$CE = A + r(P - A) \tag{14}$$

which, for $A = 0$, simplifies to:

$$CE = r\,P \tag{15}$$

**4.2.** Kontek (2009) provides separate approximations of $p = Q(r)$ for positive and negative prospects using the cumulative Beta distribution function:

$$p = Q(r) = I_r(\alpha, \beta) \tag{16}$$

where $I_r$ is the regularized incomplete beta function, and whose inverse relationship may be also given:

$$r = Q^{-1}(p) = I_p^{-1}(\alpha, \beta) \tag{17}$$

where $I_r^{-1}$ is the inverse regularized incomplete beta function. The approximation parameters are[6] $\alpha^+ = 2.028$ and $\beta^+ = 2.827$ for positive prospects and $\alpha^- = 1.604$ and $\beta^- = 2.091$ for negative prospects. The constant $\lambda = -1.71$ scales both positive and negative prospects similarly to Prospect Theory (cf Function (8)).

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[4] More precisely, $s = \frac{\ln(1+r)}{\ln(2)}$. This way, for a value of $r$ in the range [0, 1], $s$ takes its values from the same range.

[5] For example, the risk free component $A$ in a lottery expressed as {{50, 150}, {1 - p, p}} is 50.

[6] The parameters presented here differ slightly from those provided by Kontek (2009). This is because they were obtained using a modified method based on approximating the inverse function $r = Q^{-1}(p)$, rather than $p = Q(r)$. This method happens to be more stable, especially for data sets larger than those provided in Kahneman and Tversky's article (1992). The details are provided in another, as yet unpublished, work of this author.

**4.3.** An approximation of both parts of the relative utility function can be also given using the cumulative Kumaraswamy distribution function:

$$p = Q(r) = 1 - \left(1 - r^{\alpha}\right)^{\beta} \qquad (18)$$

This function has several advantages. Firstly, it gives a better $p = U(s)$ approximation than does the cumulative Beta distribution function (Kontek, 2009). Secondly, the inverse function can be obtained analytically:

$$r = Q^{-1}(p) = \left(1 - (1 - p)^{1/\beta}\right)^{1/\alpha} \qquad (19)$$

The approximation parameters are $\alpha^{+} = 1.761$ and $\beta^{+} = 2.993$ for positive prospects and $\alpha^{-} = 1.497$ and $\beta^{-} = 2.151$ for negative prospects. The scaling constant $\lambda$ (cf Function (8)) equals -1.71. As is shown later in this paper, the Kumaraswamy approximation expressed by Functions (18) and (19) is more useful for discrete outcomes, whereas the Beta approximation expressed by Functions (16) and (17) gives simpler solutions for continuous outcome distributions.

## 5. Calculation of Certainty Equivalent for Discrete Outcomes.

**5.1.** The methods presented in Points 2 and 3 assumed that lotteries are compared on the basis of their "values". However, this line of reasoning is not completely intuitive as the unit of "value" is not strictly defined (utils?). The present method takes a different approach in that it uses certainty equivalents instead of hypothetical values to compare lotteries. More importantly, the concept of lottery value (utility) is not even considered[7].

The certainty equivalent *CE* of a two-outcome lottery is determined directly from Functions (19) or (17) and then Function (14), because the relative certainty equivalent $r$ transforms directly to probability $p$ (and vice versa) (Refer Fig. 1, Left).

The procedure is slightly more involved for lotteries with more than two outcomes. The lottery is replaced by an equivalent, single-prize lottery with an equivalent probability $p_E$ of winning that one prize. The procedure then continues as in the former case.

**5.2.** This result will be derived more formally later in this paper. For now, a simple example which illustrates the method by calculating the expected value of the lottery will be given. Let us assume a lottery offers $x_1$ = \$100 with a probability of $p_1$ = 0.5, $x_2$ = \$60 with a probability of $p_2$ = 0.3 and $x_3$ = \$0 with a probability $p_3$ = 0.2. The expected value of the lottery can be obtained directly from Function (1):

$$EX = x_1 p_1 + x_2 p_2 + x_3 p_3 = 0.5 \times 100 + 0.3 \times 60 + 0.2 \times 0 = 68 \qquad (20)$$

[7] We do, however, use the "relative utility" term. This will be explained later.

This lottery will now be replaced by an equivalent, single-prize lottery. Let the outcomes \$100, \$60 and \$0 be the expected values of other independent lotteries with a single prize of \$100. It can easily be seen that the probabilities of winning the given outcomes are $p_1'$ = 100%, $p_2'$ = 60% and $p_3'$ = 0% respectively. Because the lotteries are independent, the joint probability of winning the main prize of \$100 is:

$$p = p_1 p_1' + p_2 p_2' + p_3 p_3' = 0.5 \times 1.0 + 0.3 \times 0.6 + 0.2 \times 0.0 = 0.68 \tag{21}$$

The three-outcome lottery can therefore be replaced by an equivalent lottery with a major (and sole) prize of \$100 and a 68% probability of winning it. The expected value of the equivalent lottery is thus:

$$EX = 0.68 \times 100 = 68 \tag{22}$$

which (not surprisingly) is exactly the same as of that the lottery under consideration in Function (20).

**5.3.** Let us return to certainty equivalent considerations and analyze the above example in a more general way. Let us assume that those outcomes which are members of the set $X$ are non-negative and that the set of relative outcomes $R$ consists of real numbers in the range [0, 1]. Let us define an outcome vector $x = \{x_1, x_2, \dots , x_n\}$ such that $x_1, x_2, \dots, x_n \in X$ and a corresponding relative outcome vector $r = \{r_1, r_2, \dots , r_n\}$ such that $r_1, r_2, \dots, r_n \in R$. The relationship between the outcome vector $x$ and the relative outcome vector $r$ is analogous to Function (12):

$$r = \frac{x - A}{P - A} \tag{23}$$

where $A = Min(x)$ is the risk free component of the lottery and $P = Max(x)$ is its maximum outcome. The relationship described by Function (23) ensures that there is at least one relative outcome $r_i$ equal to 1 and at least one relative outcome equal to 0.[8] Let us finally define a vector of lottery probabilities $p = \{p_1, p_2 \dots p_n\}$ such that $\sum_{i=1}^{n} p_i = 1$, where $p_i$ is the probability of relative outcome $r_i$. The normalized lottery under consideration is therefore defined by the vector $\{r, p\}$.

Let the relative outcomes $r_i$ be the certainty equivalents of other independent lotteries with a single prize expressed in relative terms as 1 and with probabilities of winning it of $p_i'$.

[8] Which outcomes these are, however, is irrelevant because the order of calculations is unimportant in the method herein presented (in contrast with CPT). It is also irrelevant how many relative outcomes take value of 0, 1 or of any other value in the range [0, 1]. This additionally distinguishes the present methodology from CPT.

These lotteries will be called primary lotteries[9] and the probabilities $p_i'$ the primary probabilities of the lottery under consideration. The primary lotteries described as {{0, 1}, {1 - $p_i'$, $p_i'$}} are therefore equivalent to {{$r_i$}, {1}}. The primary probabilities can be easily obtained because $p_i' = Q(r_i)$ what allows the primary probability vector $p' = \{p_1', p_2', \ldots, p_n'\}$ to be determined. Assuming the primary lotteries are independent, the joint probability of winning the main prize is given similarly to (21) by:

$$p_E = \sum_{i=1}^{n} p_i p_i' = p.p' \qquad (24)$$

It should be clear from the above that the lottery {{0, 1}, {1- $p_E$, $p_E$}} having only two outcomes 0 and 1 is equivalent to the lottery $\{r, p\}$ having outcomes $r$ in the range [0, 1]. This is why the probability $p_E$ is called the equivalent probability of the lottery considered. Taking into account the definition of the primary probability vector $p'$ stated in the former paragraph the function (24) can alternatively be presented as:

$$p_E = \sum_{i=1}^{n} p_i Q(r_i) \qquad (25)$$

which (quite surprisingly) resembles the manner in which Expected Utility Theory determines the lottery value. This explains the introduction of the term "relative utility function" for the curve $p = Q(r)$, despite the fact that the concept of utility is not required for the solution. This also demonstrates that the "relative utility" is expressed in terms of probability. This means that all further considerations can proceed on the basis of classical probability theory without introducing any additional concepts.

Having obtained the equivalent probability $p_E$ from Function (25), the relative equivalent $CE_r$ of the lottery can be obtained by applying Functions (17) or (19):

$$CE_r = Q^{-1}(p_E) = Q^{-1}(p.p') = Q^{-1}\left(\sum_{i=1}^{n} p_i p_i'\right) = Q^{-1}\left(\sum_{i=1}^{n} p_i Q(r_i)\right) \qquad (26)$$

Consequently, and analogously to Function (14), the certainty equivalent, expressed as a real, rather than a relative, value can be calculated as:

$$CE = A + CE_r(P - A) \qquad (27)$$

which is the final result of the methodology introduced here.

**5.4.** Let us now analyze the issue of stochastic dominance, discussed in Point 3.3., which was a source of severe problems for the original Prospect Theory. To recapitulate, it was this issue that led to the concept of the probability weighting function for cumulative

[9] Like primary elections.

probabilities adopted by Tversky and Kahneman (1992) in their Cumulative Prospect Theory. As will be demonstrated, this problem does not arise in our methodology. Let us assume that one of the relative outcomes of $r_{i,}$ which has a probability of $p_{i,}$ is presented as $k$ outcomes of $r_i$, each having a probability of $p_i$ / $k$. For simplification, let this outcome be $r_n$, the last. Applying Function (25) gives:

$$p_E = \sum_{i=1}^{n-1} p_i Q(r_i) + \sum_{j=1}^{k} \frac{p_n}{k} Q(r_n) = \sum_{i=1}^{n-1} p_i Q(r_i) + p_n Q(r_n) \sum_{j=1}^{k} \frac{1}{k} = \sum_{i=1}^{n-1} p_i Q(r_i) + p_n Q(r_n) = \sum_{i=1}^{n} p_i Q(r_i) \quad (28)$$

The above reasoning can obviously be generalized to all $r_i$. This means that the way in which a lottery is represented has no impact on its equivalent probability $p_E$, and consequently, its certainty equivalent *CE*.

**5.5.** This conclusion should be apparent, once we consider that the methodology is based on classical probability calculus and neither adopts any weighting of probabilities nor sums their values. As can be seen from Functions (24) – (26), the calculations can be performed whatever the order of outcomes (i.e. unlike CPT, it is not necessary to rank outcomes by value). Similarly, it is not necessary to derive the cumulative probability distribution function, perform non-linear transformations thereon and return to increments of weights. As a result, the value of a lottery does not depend on how it is represented and no "tricks" are required to get around this problem. Saying it using other words, as our methodology does not introduce any artificial concepts it does not require any special solutions in order to resolve arisen difficulties.

The obtained result has another important feature. The form of Functions (25) and (26) strongly resembles the method of lottery valuation postulated by Expected Utility Theory. The difference is that there is a relative utility function *Q* instead of a utility function *U*. Additionally, the methodology herein presented requires no probability weighting function to calculate certainty equivalents of lotteries. This method therefore signals a complete departure from Prospect Theory and a return to the classical approach.

## 6. Allais Paradox.

**6.1.** One application of the methodology introduced above is demonstrated in this section. We will consider one of the classical problems illustrating the irrationality of human decision making, viz. the Allais paradox. The "Prospect Theory" paper presented this as follows:

*Problem 1: Choose between*

A: 2,500 with a probability of .33,    B: 2,400 with a probability of 1

2,400 with a probability of .66

0 with a probability of .01

*Problem 2: Choose between*

C: 2,500 with a probability of .33 D: 2,400 with a probability of .34

0 with a probability of .67 0 with a probability of .66

Experimental results consistently reveal that most people choose option B in Problem 1 and option C in Problem 2. Expected Utility Theory, per contra, predicts that people would choose either A and C or B and D. It should be noted that this paradox, despite being presented together with experimental results in the "Prospect Theory" paper, was not properly resolved at the time. This is because the original 1979 theory only provided hypothetical shapes for the value and decision weighting functions. CPT later provided approximations for both curves but not a detailed solution to the Allais paradox. We will offer a solution based on the relative utility function in the next part of this section. This will then be compared with the CPT solution.

**6.2.** Somewhat unusually, we begin by analyzing options C and D, noticing that the stated preference of C over D is not in the least surprising. The expected values of options C and D are 825 and 816 respectively. Thus, even on the grounds of classical considerations, option C is a better choice than D and this is confirmed experimentally. Calculating certainty equivalents leads to the same conclusion. Because options C and D are single-prize lotteries, their certainty equivalents can be obtained directly from Functions (19) and (15). These are $CE_C = 2500\ Q^{-1}(0.33) = 768.4$ for option C, and $CE_D = 2400\ Q^{-1}(0.34) = 752.2$ for option D. The certainty equivalent of option C is greater than of option D by a value of 16.2, so it is a better choice on the grounds of behavioral considerations as well.

The situation looks different when considering the choice between options A and B. The expected value of option A (2409) is slightly greater than that of option B (2400). Classical probability theory would therefore predict a preference for A. An additional "rational" argument for A is that options A and B arise by adding the same component (2400, 0.66) to both C and D. So, if C is a better choice than D, then A should be a better choice than B. This, however, is contradicted by the experimental results. This anomaly can be explained by assuming that the 1% probability of winning nothing in option A revises its certainty equivalent substantially downward, making it a much less attractive choice than the expected value calculation would have it appear.

The calculations confirming this assumption are as follows. The outcome vector $x$ = {2500, 2400, 0}[10] in option A is replaced by a relative outcome vector $r$ = {1, 0.960, 0} for which a primary probability vector $p'$ = {1, 0.9997, 0} can be obtained. The lottery probability vector is $p$ = {0.33, 0.66, 0.01}, so the equivalent probability is $p_E = p.p'$ = 0.990. The relative certainty equivalent is therefore $CE_r$ = 0.871 and the certainty equivalent is $CE$ = 2176.9, which is significantly lower than the expected value of 2409 previously calculated. Option B has a certainty equivalent of 2400 and is therefore preferred despite its lower expected value. The Allais paradox is solved.

**6.3.** The solution provided in point 6.2. will now be compared with the CPT solution. Once again, we start with an analysis of options C and D whose values can be calculated without transforming probabilities to a cumulative form. Applying Function (7), option C has a value of $V(\text{C}) = 2500^{0.88}\ W(0.33) = 326.7$, and option D has a value of $V(\text{D}) = 2400^{0.88}\ W(0.34) = 320.1$. This indicates a preference for C over D, but because the unit of value of V is undefined, the comparison is not completely satisfactory[11]. The following operations have to be performed before options A and B can be compared. For a ranked outcome vector $x$ = {2500, 2400, 0}[12] and an outcome value vector $u$ = {977.7, 943.2, 0}, a lottery probability vector $p$ = {0.33, 0.66, 0.01} can be defined, along with its cumulative representation $cp$ = {0.33, 0.99, 1}. A weighted cumulative probability vector $w$ = {0.334, 0.912, 1} is then calculated, based on which a vector of increments of weighted cumulative probabilities $\Delta w$ = {0.334, 0.577, 0.088} can be obtained. The value of option A is $V(\text{A}) = u.\Delta w = 871.3$. The value of option B has to be calculated despite the fact that it gives a certain outcome. The value of option B is $V(\text{B}) = 2400^{0.88} = 943.2$. Option B is therefore preferable to option A[13].

**6.4.** This analysis of the Allais paradox shows that calculating certainty equivalents using the relative utility function leads to similar conclusions as CPT, but more importantly it leads to full conformance with the decisions stated in experiments. Quite clearly, using certainty equivalents to compare lotteries is more natural and intuitive than using hypothetical

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[10] The order of outcomes is irrelevant here.

[11] It is hard to state what a difference of 6.6 in value exactly means. Comparing certainty equivalents would obviously be more intuitive, but some further steps had to be taken. Using (11) we obtain $CE_C$ = 719.6 and $CE_D$ = 703.0, which values are significantly lower than those obtained using the relative utility function. However, the difference between both equivalents is 16.6 in favor of option C (which is actually very close to the 16.2 obtained with the relative utility function). This confirms the advantage of option C over option D.

[12] Outcomes have to be ranked by value.

[13] Once again the difference of 71.9 in value does not explain much. But as before, options can be compared by their certainty equivalents. The certainty equivalent $CE_A$ = 2193.1 for option A is lower than the 2400 for option B. This confirms the fact that B is a better choice than A but illustrates the preference in a more intuitive way.

lottery "values". More natural and intuitive is likewise the method of achieving the final result using the relative utility function[14].

## 7. Calculating Certainty Equivalents for Negative and Mixed Lotteries.

**7.1.** The process of deriving solutions for negative lotteries is analogous to that for positive ones (refer Point 5.3.), the only difference being the negative values of the outcomes. The risk free component shall therefore be defined as $A = Max(x)$, and the maximum outcome (in absolute terms) as $P = Min(x)$. As the numerator and denominator in Function (23) are both negative, the variable $r$ is positive, which leads to exactly the same solutions presented in Functions (24) - (27). The relative utility function $Q$ simply needs to be parameterized for losses when calculating certainty equivalents for negative lotteries (see 4.2 and 4.3).

**7.2.** Once a method for calculating positive and negative lotteries has been devised, a generalized solution for mixed lotteries can easily be obtained. The only thing that needs to be added is a coefficient $\lambda$ to scale gains and losses. Analogously to (8) we can state:

$$p = \lambda Q(r) \tag{29}$$

where $\lambda$ is the scaling constant whose value is -1.71 for losses and 1.0 for gains. The function $Q(r)$ function takes the form $Q^{-}(r)$ for losses and $Q^{+}(r)$ for gains, with the respective parameterization given in section 4.2. The certainty equivalent can therefore be obtained using the following expression:

$$CE_r = Q^{-1}\left(\lambda\sum_{j=1}^{m} p_j Q^{-}(r_j) + \sum_{i=1}^{n} p_i Q^{+}(r_i)\right) \tag{30}$$

where the subscripts $j...m$ apply to losses and $i…n$ to gains. The choice of either $Q^{+}(r)$ or $Q^{-}(r)$ as the inverse function depends on the sign of the result in parentheses.

## 8. Fair / Unfair Symmetric Bet.

**8.1.** The "Prospect Theory" paper (Kahneman, Tversky, 1979) refers to research conducted by Williams (1966), who stated that the subjects he examined were indifferent between prospects (100, 0.65; -100, 0.35) and (0). Similarly to the Allais paradox, this problem was not resolved in the paper describing Prospect Theory.

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[14] If somebody has doubts regarding this statement then he/she should give a simple explanation why the value of $\Delta w_2 = 0.577$ instead of probability $p_2 = 0.66$ is used for lottery valuation in CPT and why $\Delta w$ would be different if the same probability was assigned to another outcome.

**8.2.** Let us find a general solution to the problem of which conditions have to be met for lottery {{-*x*, *x*}, {1 - *p*, *p*}} to be equivalent to lottery {{0}, {1}}. The relative utility function can solve this instantly. The following relationship must hold:

$$-\lambda(1-p)Q^{-}(1) = pQ^{+}(1) \tag{31}$$

hence

$$-\lambda(1-p) = p \tag{32}$$

or (remember that $\lambda$ = -1.71)

$$p = \frac{\lambda}{\lambda - 1} = 0.63 \tag{33}$$

which is close to the probability of 0.65 that Williams obtained experimentally. Furthermore, this solution is independent of outcome *x*.

**8.3.** Let us now find a similar solution using Prospect Theory. The following relationship must hold:

$$-\lambda x^{\alpha} W^{-}(1-p) = x^{\alpha} W^{+}(p) \tag{34}$$

which leads to:

$$-\lambda \frac{(1-p)^{\delta}}{\left((1-p)^{\delta} + p^{\delta}\right)^{\frac{1}{\delta}}} = \frac{p^{\gamma}}{\left(p^{\gamma} + (1-p)^{\gamma}\right)^{\frac{1}{\gamma}}} \tag{35}$$

This result of Function (35) is likewise independent of outcome *x*, but can only be solved numerically. Substituting the CPT parameters (refer Point 3.5.) gives *p* = 0.79, which differs significantly from the experimental value to which Prospect Theory referred.

**8.4.** The solution achieved using the relative utility function is not only simpler but, first of all, predicts the experimental results much better than does the solution provided by CPT.

## 9. Calculating Certainty Equivalents for Outcomes with Continuous Distribution.

**9.1.** The results obtained for discrete outcomes can now be easily generalized to cover continuous distributions. Analogously to Function (26), the relative certainty equivalent $CE_r$ is given by the formula:

$$CE_r = Q^{-1}\left(\int_0^1 P(r)Q(r)dr\right) \tag{36}$$

where *P*(*r*) is the density distribution function of variable *r* over the range [0,1].

**9.2.** The certainty equivalent for the uniform distribution *P*(*r*) = 1 over the range [0,1] can be found with the aid of Function (36). The equivalent probability is given by:

$$p_E = \int_0^1 \left(1 - \left(1 - r^{\alpha}\right)^{\beta}\right) dr = 1 - \frac{\Gamma\left(1 + \frac{1}{\alpha}\right)\Gamma(\beta + 1)}{\Gamma\left(\beta + \frac{1}{\alpha} + 1\right)} \tag{37}$$

where $\Gamma$ is a Gamma function. The certainty equivalent can now be computed from:

$$CE_r = Q^{-1}(p_E) = \left(1 - \left(\frac{\Gamma\left(1 + \frac{1}{\alpha}\right)\Gamma(\beta + 1)}{\Gamma\left(\beta + \frac{1}{\alpha} + 1\right)}\right)^{1/\beta}\right)^{1/\alpha} \tag{38}$$

The result for positive lotteries is:

$$CE_r^{+} = Q^{-1}(0.582) = 0.458 \tag{39}$$

and the result for negative lotteries is:

$$CE_r^{-} = Q^{-1}(0.566) = 0.469 \tag{40}$$

These results show that the certainty equivalent of a lottery with evenly distributed outcomes is located somewhere halfway between the certainty equivalent (~0.41) and the expected value (0.5) of a discrete lottery {{0, 1}, {0.5, 0.5}}.

**9.3.** One obvious downside to Function (38) is its relative complexity[15]. This may mean that the Kumaraswamy approximation is not best suited to analyzing continuous distributions. The situation brightens up considerably once a cumulative Beta distribution function is used to approximate the relative utility function. Calculating the equivalent probability for a uniform distribution of outcomes in this case leads to the simple result:

$$p_E = \int_0^1 I_r(\alpha, \beta) dr = \frac{\beta}{\alpha + \beta} \tag{41}$$

The relative certainty equivalent is therefore given by:

$$CE_r = Q^{-1}(p_{CE}) = I^{-1}_{\frac{\beta}{\alpha+\beta}}(\alpha, \beta) \tag{42}$$

This gives the following equivalents for positive and negative lotteries respectively:

$$CE_r^{+} = Q^{-1}(0.582) = 0.455 \tag{43}$$

$$CE_r^{-} = Q^{-1}(0.566) = 0.467 \tag{44}$$

Not too surprisingly, the equivalent probabilities obtained are exactly the same as those in Functions (39) and (40), although the relative certainty equivalents differ slightly.

[15] However it is still an analytical solution. The probability weighting function in the form presented either by Kahneman and Tversky or Prelec does not have a closed-forms integral in the range [0, 1]. This makes it virtually impossible to achieve any analytical solutions with Prospect Theory.

**9.4.** The above approximation can be used to analyze a much more complex case, viz. an outcome distribution $r$ described by a Beta distribution $BD$ with parameters $\gamma$ and $\delta$:

$$P(r) = BD(\gamma, \delta, r) = \frac{r^{\gamma-1}(1-r)^{\delta-1}}{B(\gamma, \delta)} \qquad (45)$$

where $B$ is the Beta function. The equivalent probability is then given by:

$$p_E = \int_0^1 BD(\gamma, \delta, r) I_r(\alpha, \beta) dr = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+\gamma)\Gamma(\gamma+\delta)}{\Gamma(\beta)\Gamma(\gamma)} {}_3\widetilde{F}_2(\alpha, 1-\beta, \alpha+\gamma; \alpha+1, \alpha+\gamma+\delta; 1) \qquad (46)$$

where ${}_3\widetilde{F}_2(a,b,c;d,e;f)$ is a regularized hypergeometric function. It can be verified that Function (46) simplifies to Function (41) for a Beta distribution with $\gamma = 1$ and $\delta = 1$, i.e. for an uniform outcome distribution.

Figure 2 shows examples of single–mode outcome distributions presented on the same graph together with the relative utility function, the outcome distribution mean, and the resulting equivalent probability and relative certainty equivalent values. This type of presentation is made possible by the relative utility function and the outcome distribution both being in the range [0, 1] and both being expressed in terms of probability. Additionally, both the outcome distribution mean and relative certainty equivalent are expressed as the fractions of the main prize of the lottery while the equivalent probability is expressed in terms of the probability of winning it. This indicates that the methodology introduced has a highly unified and integrated approach to problem solving.

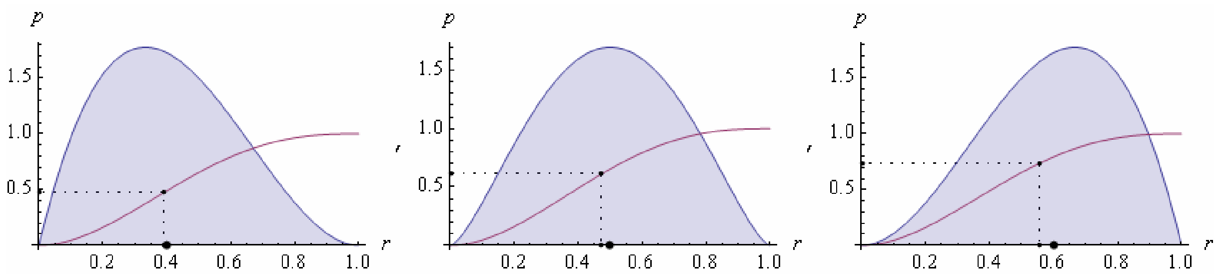


Fig. 2. Examples of Beta distributions (shaded) for $\gamma = 2$ and $\delta = 3$ (left), $\gamma = 2.5$ and $\delta = 2.5$ (center), and $\gamma = 3$ and $\delta = 2$ (right). The corresponding distribution means 0.4, 0.5 and 0.6 are shown as large dots on the $r$ axis. The S-shaped curve is the aspiration / relative utility function. The calculated relative certainty equivalents ($CE_r$) 0.391, 0.473 and 0.556 for equivalent probabilities ($p_E$) 0.476, 0.612 and 0.736 respectively are shown as small dots on the $r$ and $p$ axes.

A Beta distribution with parameters $\gamma$ and $\delta$ less than 1 has an interesting shape which allows the certainty equivalents for quasi-discrete outcomes to be examined. Fig. 3 shows cases where most of the outcomes are close to 0 (left), close to the main prize (right) and evenly divided between the two (center).

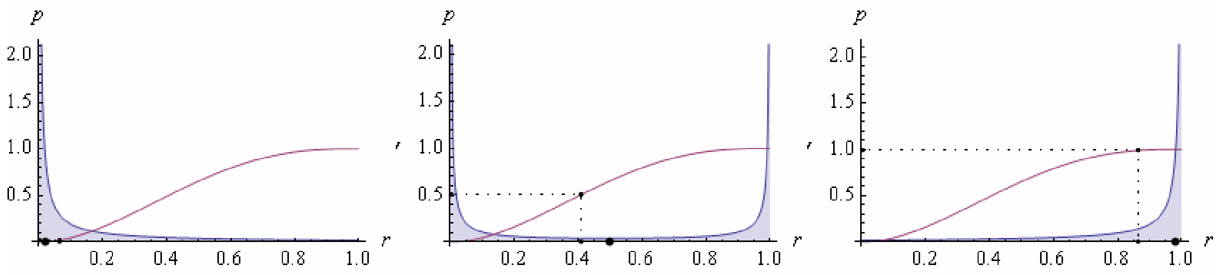


Fig.3 Examples of Beta distributions (shaded) $r$ for $\gamma = 0.02$ and $\delta = 1$ (left), $\gamma = 0.02$ and $\delta = 0.02$ (center), and $\gamma = 1$ and $\delta = 0.02$ (right). The corresponding distribution means 0.0196, 0.5 and 0.9804 are shown as large dots on the $r$ axis. The S-shaped curve is the aspiration / relative utility function. The calculated relative certainty equivalents ($CE_r$) 0.066, 0.408 and 0.865 for equivalent probabilities ($p_E$) 0.020, 0.504 and 0.988 respectively are shown as small dots on the $r$ and $p$ axes.

These results can be compared with discrete lotteries. The center diagram in Fig. 3, for example, resembles a lottery {{0, 1}, {0.5, 0.5}} and its certainty equivalent (0.408) is practically identical to that of a discrete lottery (0.406).

**9.5.** Solution (46) may now be extended to cover the more complex case of lottery outcomes being described by a mix of Beta distributions[16]:

$$P(r) = \sum_{i=1}^{n} a_i BD_i(\gamma_i, \delta_i, r) \tag{47}$$

where $BD_i$ are consecutive Beta distributions with $a_i$ as their weights, each in the range [0, 1] and all fulfilling $\sum_{i=1}^{n} a_i = 1$, which create a vector of weights $a = \{a_1, a_2, \ldots, a_n\}$. The property of additivity stated during derivation of the equivalent probability $p_E$ formula (Refer 25) results in:

$$p_E = \int_0^1 \left( \sum_{i=1}^{n} a_i BD_i(\gamma_i, \delta_i, r) \right) I_r(\alpha, \beta) dr = \sum_{i=1}^{n} a_i \int_0^1 BD_i(\gamma_i, \delta_i, r) I_r(\alpha, \beta) dr = \sum_{i=1}^{n} a_i p_{ei} = a.p_e \tag{48}$$

where each individual equivalent probability $p_{ei}$ is calculated by applying Function (46) and all $p_{ei}$ form a vector $p_e = \{p_{e1}, p_{e2} \ldots p_{en}\}$. The result (48) shows that the equivalent probability $p_E$ of an outcome distribution which can be described by a mix of Beta distributions is the mean value of their individual equivalent probabilities. Fig. 4 shows several (rather random) examples of mixed Beta distributions together with their resulting certainty equivalent values.

[16] We talk here about a shape described by a linear combination of Beta distributions. We do not have in mind a mixture of random variables having Beta distribution which would result a different shape.

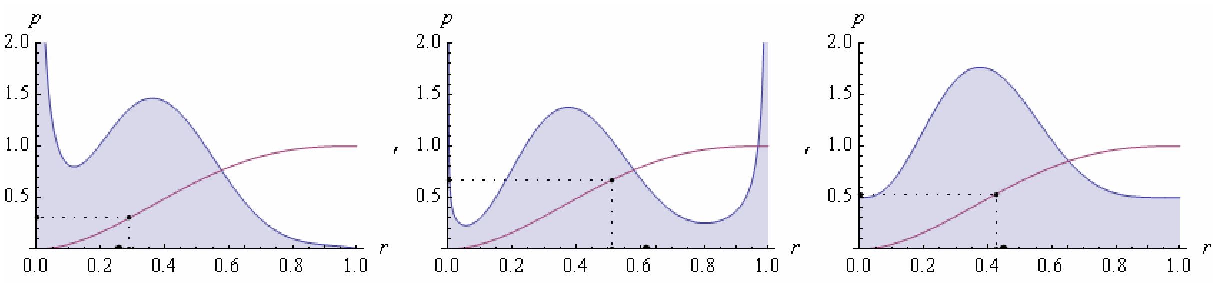


Fig.4 Examples of relative outcome $r$ distributions described as a mix of Beta distributions. A Beta distribution with parameters $\gamma_1 = 4$ and $\delta_1 = 6$ is evenly mixed with another Beta distribution of $\gamma_2 = 0.2$ and $\delta_2 = 1.5$ (left), $\gamma_2 = 0.5$ and $\delta_2 = 0.1$ (center), and $\gamma_2 = 1$ and $\delta_2 = 1$ (right). The corresponding distribution means 0.259, 0.617 and 0.450 are presented as large dots on the $r$ axis. The S-shaped curve is the aspiration / relative utility function. The relative certainty equivalents ($CE_r$) 0.290, 0.512 and 0.426 calculated for equivalent probabilities $p_E$ of 0.302, 0.673 and 0.532 respectively are shown as small dots on the $r$ and $p$ axes.

**9.6.** Important to note the result (48) is of a very general nature, as it may concern a mix of discrete, continuous and both discrete and continuous outcome distributions, including those which are not Beta ones. This conclusions together with the examples illustrated in Fig. 4 show that the approach introduced in this section is extremely universal and enables analytical solutions to be found, even for extremely complex cases of outcomes.

## 10. Calculating Certainty Equivalents for Mixed Lotteries with Continuous Outcome Distributions

**10.1.** Calculating certainty equivalents for a mixed lottery is analogous to Function (30):

$$CE_r = Q^{-1}\left(\lambda\int_0^1 P(r)Q^-(r)dr + \int_0^1 P(r)Q^+(r)dr\right) \tag{49}$$

where the integrals for positive and negative lotteries are calculated separately.

**10.2.** Let us calculate the certainty equivalent for a lottery with an uniform outcome distribution in the range [-100, 100]. Applying Function (41) gives:

$$p_E = \frac{\lambda}{2}\frac{\beta^-}{\alpha^- + \beta^-} + \frac{1}{2}\frac{\beta^+}{\alpha^+ + \beta^+} = -0.193 \tag{50}$$

Because the value obtained is negative, an inverse function $Q^{-1}$ with parameters for losses[17] has to be used and the argument needs to be set to $p_E / \lambda = 0.113$ (and not $p_E$!):

$$CE_r = I^{-1}_{\frac{p_E}{\lambda}}\left(\alpha^-, \beta^-\right) = 0.145 \tag{51}$$

[17] Interesting to note that instead of using the inverse regularized incomplete beta function in order to find $Q^{-1}$ as in (51) one can always switch to the Kumaraswamy form (19), which might be easier to calculate.

The certainty equivalent is therefore $CE = -100\ CE_r = -14.5$. Solving this problem using Prospect Theory is a task best left to the most ardent proponents of that theory[18].

## 11. Relative Utility Theory.

**11.1.** The material presented so far lays the foundations for Relative Utility Theory – a new theory of decision making under conditions of risk. This claim is made on the basis of the derivation of a generic method for calculating certainty equivalents of multi-outcome lotteries. The solutions presented in this paper apply to lotteries with positive, negative and mixed outcomes as well as to those with discrete and continuous outcome distributions. Moreover, parameterizations have been provided that enable the theory to be put to use in practical applications, as is evidenced by the solutions to several (some of them hitherto intractable) problems that were presented in this paper. These include the Allais paradox – one of the most classical examples of how the Expected Utility Theory fails to predict or explain the experimental data.

**11.2.** The main distinguishing feature of the new theory lies in its performance of calculations on relative, rather than absolute, outcome values, which was the underlying assumption of the historical theories presented in Point 2 of this paper. The first departure from such an approach was made by Prospect Theory, which utilized the idea of gains and losses around a reference point. This followed from the assumption that people adapt to their current level of wealth and that these adaptations can be described by certain mental operations performed during the Editing phase, i.e. before a lottery has been properly valued. It is however important to note that gains and losses were still represented as specific monetary outcomes, despite often being expressed in a relative form[19].

The concept of a value function for gains and losses was, however, not sufficient to fully interpret other phenomena encountered in experiments on decision making under conditions of risk. This necessitated the inclusion of the concept of probability distortion. This was reflected by a probability weighting function, which thus became one of the key planks of Prospect Theory, especially its Cumulative version.

Prospect Theory however failed to cater for another, particularly important mental operation, viz. attention focus. Taking attention focus into account leads to some vital conclusions (Kontek, 2009). First and foremost, every outcome of a lottery is perceived as

[18] Presenting the solution would take too much place here. Besides due to the approach complexity we are not quite sure about the result, which can, anyhow, be computed only numerically. We assume it is -24.1.

[19] For example, people usually say “I made 10% on the stock market, but I lost 5% on property”, rather than giving specific monetary outcomes.

being relative to a certain value on which attention is focused, usually the maximum outcome. This leads to a relative utility function determined for relative outcome values. For this reason, a transformation of absolute outcomes to relative ones, as effected by Function (23), is essential to the present theory. Another, extremely important implication of adopting attention focus process is a complete rejection of any probability weighting function to explain the experimental results.

**11.3.** The next crucial distinguishing feature of the theory is that the aspiration / relative utility function is expressed in terms of probability. This means that multi-outcome lotteries can be analyzed as independent primary lotteries on the grounds of classical probability theory. The concept of utility does not even have to be introduced and the term "relative utility" merely serves to recall historical discourse on the subject.

**11.4.** The final distinguishing feature of the present theory is that it utilizes the certainty equivalents of lotteries, rather than their hypothetical values, to compare them. This seems a very intuitive and easy formula to accept, especially in view of the failure of the theories so far put forward to come up with a clear and convincing way of measuring utility.

**11.5.** The methodology presented in the paper thus leads to Relative Utility Theory, which replaces all the earlier theories on decision making under conditions of risk. This has the elegant form of Expected Utility Theory but answers the main questions raised by the various Non-Expected Utility theories[20]. All this is achieved without resorting to the concept of a probability weighting function. Understanding the Mental Transformations (Kontek, 2009) carried out during the decision making process was essential to deriving the solution presented herein.

## 12. Summary.

This paper presented a method for valuing lotteries using the aspiration / relative utility function. Lotteries with discrete and continuous outcomes were discussed, as were lotteries with positive, negative and mixed outcomes. Formulas for certainty equivalents resemble the form postulated by Expected Utility Theory. Despite not using the probability weighting function, solutions were obtained for several problems, including the Allais paradox, as well as for even more complex cases of outcome distributions described as a mix of Beta distributions. The material presented in the paper provides strong arguments for

[20] For instance, the shape of the aspiration / relative utility function fully explains the "fourfold pattern of risk attitudes" put forward by CPT (Kontek, 2009).

rejecting Prospect Theory as a model for describing human behavior and lays the foundations for Relative Utility Theory – a new theory of decision making under conditions of risk.

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