Expected Utility Theory with Probability Grids

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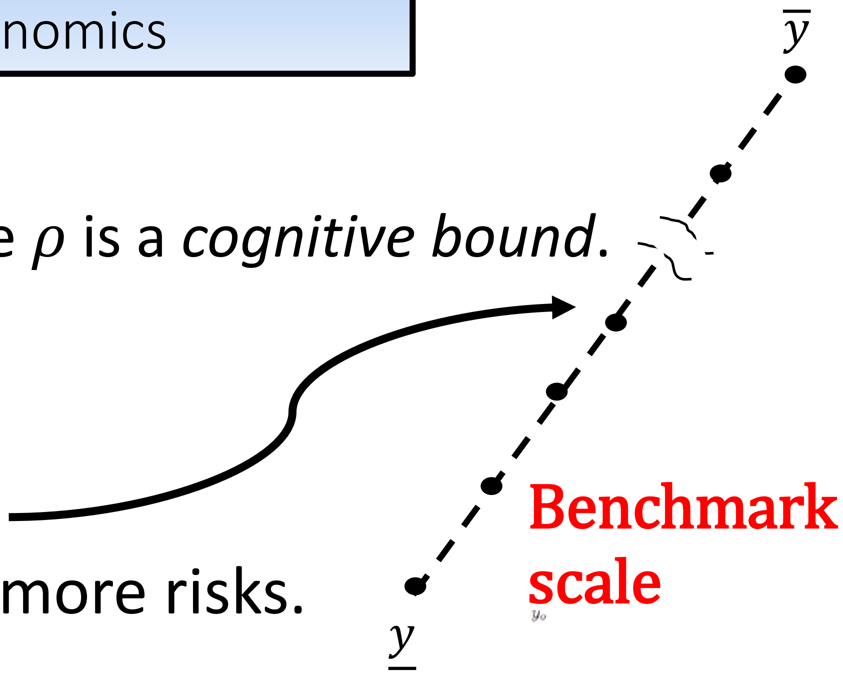
- Bounded probability set $\Pi_\rho = \{\frac{t}{10^\rho} : t = 0,1,\ldots,10^\rho\}$, where ρ is a *cognitive bound*.

- Two types of use of probability, to be separated:
  - Measurement of "utility" from a pure alternative y
  - Extension of measured utilities to lotteries involving more risks.



- Classical EU theory is the case with no cognitive bounds, i.e., $\Pi_\infty = \cup_{k\geq 0} \Pi_k$.
- Our focus is incomparabilities caused by cognitive bounds.

- The new theory is applied to the Allais paradox.
- We connect measurement of "utility" with H. Simon's satisficing/aspiration.

My three research projects

- Inductive Game Theory
  - A person constructs his view (understanding) on a social situation surrounding him, from his past experiences; Kaneko-Matsui ('99), Kaneko-Kline ('08).

- Epistemic Logic and Decision Making
  - Logical inferences and decision making; Kaneko-Nagashima ('96), Kaneko ('03).

- Social Justice
  - Social contract foundation for the Nash social welfare function (Kaneko-Nakamura ('79)).
  - Applications, e.g., income taxation.

- Some other projects such as rental housing markets with indivisibilities.

Utility Theory and Bounded Rationality

- "Utility" is a basic component for all the above projects.
- Expected utility (EU) theory is central in utility theories.

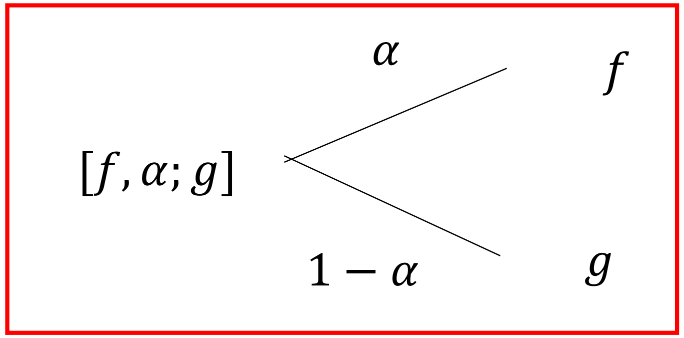
- "Bounded (and/or unbounded) rationality" is crucially relevant for those projects.
- The concept of bounded rationality has many aspects, e.g., an individual view on society should be simple, logical inference is bounded, etc., etc.

- How about (present) EU theory?

  - Preferences are assumed to exist in the mind of the decision maker (ontological assumption);
  - Free use of probability values;
  - Representation theorem is targeted – which, preferences or utility, is really the target?

Classical EU Theory

- $L(X)$ – the set of probability distributions (with finite supports) over the pure alternative set X;
- $\succsim$: a preference relation over $L(X)$.



Axiom NM1: (1) $\succsim$ is complete; (2) $\succsim$ is transitive.

Axiom NM2 (Intermediate value): if $f \succ h \succ g$, there is a compound lottery $[f, \alpha; g] \sim h$.

Axiom NM3: (Independence): (1): $f \sim g \Longrightarrow [f, \alpha; h] \sim [g, \alpha; h]$; (2): $f \succ g \Longrightarrow [f, \alpha; h] \succ [g, \alpha; h]$.

Theorem EU: $\succsim$ satisfies NM1-NM3 $\Leftrightarrow$ $\exists u: L(X) \rightarrow R$ s.t. $\forall f, g \in L(X)$ and $\alpha \in [0,1]$,

(1) $f \succsim g \Leftrightarrow u(f) \geq u(g)$; and (2) $u([f, \alpha; g]) = \alpha u(f) + (1-\alpha) u(g)$.

- Existence of $\succsim$ is assumed, free use of probability values is assumed;
- We illustrate these problems using the Allais paradox.

The Allais Paradox

An example from Kahneman-Tsversky ('79).

- 95 people are asked to choose one from each of the following choice problems:

$a = \left[4000, \frac{8}{10};\ 0, \frac{2}{10}\right]$ (20%) *or* $b = 3000$ (80%)

and

$c = [4000, \frac{2}{10};\ 0, \frac{2}{10}]$ (65%) *or* $d = [3000, \frac{25}{100};\ 0, \frac{75}{100}]$ (35%)

Modal case ($a \prec b$) & ($c \succ d$): Contradiction to EU theory:

$$\frac{8}{10}u(4000) + \frac{2}{10}u(0) < \frac{10}{10}u(3000) + \frac{0}{10}u(0)$$

$$\frac{2}{10}u(4000) + \frac{8}{10}u(0) > \frac{25}{100}u(3000) + \frac{75}{100}u(0).$$

	c: 65%	*d:* 35%
a: 20%	EU	paradox
b: 80%	paradox	EU

The opposite inequality of the second is obtained from the first by dividing by 4 ($u(0) = 0$).

A more basic derivation of the Paradox

Let $\underline{y} = 0$; $y = 3000\ (= b)$; $\overline{y} = 4000$ - - ($\underline{y}$, $\overline{y}$: benchmarks)

- **Measurement**: Find a probability λ so that $y \sim [\overline{y}, \lambda; \underline{y}]$ $(=[\overline{y}, \lambda; \overline{y}, 1-\lambda])$,
  - y is measured by the benchmark scale, e.g., $y \sim [\overline{y}, \frac{85}{100}; \underline{y}]$.

- **Extension**: Measurement is extended to $[y, \lambda; \underline{y}]$ such as $d = [y, \frac{25}{100}; \underline{y}]$.
  - EU allows to substitute $[\overline{y}, \frac{85}{100}; \underline{y}]$ for y in d.

Then, $d \sim [\overline{y}, \frac{2125}{10^4}; \underline{y}] \succ [\overline{y}, \frac{20}{10^2}; \underline{y}] = c$.

- This requires too much precision; Recall: level of significance in statistics is $\frac{5}{100}$ or $\frac{1}{100}$.
- We introduce a cognitive bound on available probabilities.

Two Steps to determine preferences for $\langle \succsim^0, \succsim^1, \ldots, \succsim^\rho \rangle$

- Step of measurement of utility from pure alternatives - - $\succsim^0, \succsim^1$.
  - $\underline{y}$, $\overline{y}$ are given pure alternatives called *benchmarks.*
  - for any $y \in Y$, we look for some $\lambda_y \in \Pi_\rho := \{\frac{v}{10^\rho} : v = 0,1, \ldots, 10^\rho\}$ so that

    $y \sim^1 [\overline{y}, \lambda_y; \underline{y}]$.
  - In fact, Y is defined by collecting such y's.

- Step of extension to lotteries including more probabilities over Y - - $\succsim^2, \succsim^3, \ldots, \succsim^\rho$.
  - Extension is step-by-step from lotteries including simple probabilities such as $\frac{v_2}{10^2}$,
  - then, to more complex probabilities $\frac{v_3}{10^3}$, and up to $\frac{v_\rho}{10^\rho}$.

- Our main concern is the case $\rho < \infty$.
- We consider the case $\rho = \infty$ as a reference case (classical EU theory).

Lotteries with probability grids and preference sequences

- *Cognitive bound* ρ, where $\rho < \infty$ or $\rho = \infty$
- Bounded probability set $\Pi_\rho = \{\frac{t}{10^\rho} : t = 0, 1, \dots, 10^\rho\}$,

  where $\Pi_\infty = \cup_{k\geq 0} \Pi_k$ - - the reference case as classical EU theory.
- $\Pi_k \subseteq \Pi_{k+1}$ for $k = 0, \dots,$ where $\Pi_0 = \{0, 1\}$.

- Y is the set of pure alternatives; and it is assumed to be finite set.
- $\overline{y}$ and $\underline{y}$ in Y are upper and lower *benchmarks.*

- $L_\rho(Y) := \{f : Y \to \Pi_\rho \mid \sum_{y\in Y} f(y) = 1\}$ - - the set of lotteries of at most depth ρ.
- $\mathbf{B}_\rho(\overline{y}; \underline{y}) = \{[\overline{y}, \lambda; \underline{y}] : \lambda \in \Pi_\rho\} \subseteq L_\rho(Y)$ - - the benchmark scale.

- $L_k(Y) \subseteq L_{k+1}(Y)$ for $k = 0, \dots,$ and $L_\infty(Y) = \cup_{k\geq 0} L_k(Y)$.

- A *preference sequence* $\langle \succsim^0, \succsim^1, \ldots, \succsim^\rho \rangle$ if $\rho < \infty$; and $\langle \succsim^0, \succsim^1, \ldots \rangle$ if $\rho = \infty$.

Axiom B0 Each $\succsim^k$ is transitive over $L_\rho(Y)$, i.e., $f \succsim^k g$ & $g \succsim^k h \Longrightarrow f \succsim^k h$.

- Completeness is not assumed.

- **Base Facet** $F = \left\langle Y; \overline{y}, \underline{y};\ \{\lambda_y\}_{y \in Y} \right\rangle$, where $\lambda_y \in \Pi_\rho$ for all $y \in Y$.

Axiom B1 (initial condition) For all $y \in Y$ and $\lambda \in \Pi_1$, $\overline{y} \succsim^0 [\overline{y}, \lambda; y]$ and $[y, \lambda; \underline{y}] \succsim^0 \underline{y}$.

Axiom B2 (Benchmark scale) $\lambda \geq \lambda' \Leftrightarrow [\,\overline{y}, \lambda; \underline{y}] \succsim^1 [\,\overline{y}, \lambda'; \underline{y}]$.

- Each $\succsim^1$ is complete over the benchmark scale $\mathbf{B}_\rho\left(\overline{y}; \underline{y}\right)$.

Axiom B3 (Measurement) $y \sim^1 [\,\overline{y}, \lambda_y; \underline{y}]$ for all $y \in Y$.

- Define $u: Y \to \Pi_\rho$ by $u_0(y) = \lambda_y$ for all $y \in Y$.

Lemma 2.1. Assume Axioms B0 – B3 for $\langle \succsim^k \rangle_{k=0}^{\rho}$. Then, $y \succsim^1 z \Leftrightarrow u_0(y) \geq u_0(z)$.

- Now, this $\succsim^0$ and $\succsim^1$ are extended to $\succsim^2, \dots, \succsim^\rho$. Let $k = 0,1, \dots, \rho$-1

Axiom B4 (preservation) If $f \succsim^k g$, then $f \succsim^{k+1} g$.

Axiom B5 (Extension) Let $(f_1, \dots, f_{10}) \in L_\rho(Y)^{10}$ and $(g_1, \dots, g_{10}) \in \mathbf{B}_\rho\left(\overline{y}; \underline{y}\right)$ so that $f_t \succsim^k g_t$ for $t = 1, \dots, 10$. Then $\sum \frac{1}{10} * f_t \succsim^{k+1} \sum \frac{1}{10} * g_t$ (these are assumed to be in $L_\rho(Y)$) and its strict part, and its dual.

Example: Let $Y = \{\overline{y}, y, \underline{y}\}$, $\rho = 2$, and $y \sim^1 [\overline{y}, \frac{85}{10^2}; \underline{y}]$. Consider $d = [y, \frac{25}{10^2}; \underline{y}]$.

1. $y \sim^1 [\overline{y}, \frac{85}{10^2}; \underline{y}] \succ^1 [\overline{y}, \frac{8}{10}; \underline{y}]$ by B3 and B2.

2. $[y, \frac{5}{10}; \underline{y}] \succsim^0 \underline{y}$ by B1.

3. $d = [y, \frac{25}{10^2}; \underline{y}] = \frac{2}{10} * y + \frac{1}{10} * [y, \frac{5}{10}; \underline{y}] + \frac{7}{10} * \underline{y}$

$\succsim^1 \frac{2}{10} * y + \frac{1}{10} * \underline{y} + \frac{7}{10} * \underline{y} = [y, \frac{2}{10}; \underline{y}]$ by B1 and B5.

$[y, \frac{2}{10}; \underline{y}] = \frac{2}{10} * y + \frac{8}{10} * \underline{y} \succ^2 \frac{2}{10} * [\overline{y}, \frac{8}{10}; \underline{y}] + \frac{8}{10} * \underline{y} = [\overline{y}, \frac{16}{10^2}; \underline{y}]$ by 1 and B5.

- $[\overline{y}, \frac{16}{10^2}; \underline{y}]$ is the "greatest lower bound" of d in the benchmark scale $\mathbf{B}_\rho(\overline{y}; \underline{y})$.

We look at the resulting preference relation:

- For $\rho < \infty$, we focus on the last relation $\succsim^{\rho}$ of $\langle \succsim^{0}, \dots, \succsim^{\rho} \rangle$ (on $L_{\rho}(Y)$).
- for $\rho = \infty$, we denote $\succsim^{\infty} = \cup_{k=0}^{\infty} (\succsim^{k})$ (on $L_{\infty}(Y) = \cup_{k \geq 0} L_k(Y)$).

- **Classical EU case: $\boldsymbol{\rho = \infty}$.**

Theorem 1: Assume B0 – B5 for $\langle \succsim^{0}, \succsim^{1}, \dots \rangle$. Then, for any $f, g \in L_{\infty}(Y)$,

$$f \succsim^{\infty} g \quad \Leftrightarrow u_{eu}(f) \geq u_{eu}(g),$$

where $u_{eu}(h) = \sum_{y \in Y} u_0(y) h(y)$.

- The resulting preference relation $\succsim^{\infty}$ is complete and uniquely determined.
- Each intermediate step $\succsim^{k}$ is neither complete nor unique.
- For this theorem, $\succsim^{0}$ and Axiom B1 are redundant.

- Our main interests are in the case of $\rho < \infty$.

Construction:

- B1, B2, B3 are “ontological” .
- B4, B5 generate new preferences in $\precsim_c^{k+1}$ from $\precsim_c^k$.
- B0(Transitivity) takes the transitive closure.

Theorem 2. The smallest preference sequence $\langle \precsim_c^0, \precsim_c^1, \dots, \precsim_c^\rho \rangle$ is uniquely determined by B0 – B5; for any $\langle \precsim^0, \precsim^1, \dots, \precsim^\rho \rangle$ satisfying B0 –B5,

for any $f, g \in L_\rho(Y)$ and $k = 0, \dots, \rho$, $f \precsim_c^k g \Longrightarrow f \precsim^k g$.

- $\precsim_c^\rho$ is incomplete, i.e., some lotteries are incomparable.
- We focus on $\precsim_c^\rho$, which is abbreviated as $\precsim_c$. We call this the central (preference) relation.

Measurable Domain $M(F;\rho)$

Define $M(F;\rho) = \{f \in L_\rho(Y) : f \sim_c [\bar{y}, \lambda : \underline{y}] \text{ for some } \lambda \in \Pi_\rho\}$.

- $\succsim_c$ is complete over $M(F;\rho)$ (actually the same as the eu-preferences).
- The question is how $\succsim_c$ behave over $f, g \in L_\rho(Y) - M(F;\rho)$.

We define the greatest lower bound (GLB) for $f \in L_\rho(Y)$ by

- $\underline{\lambda}_f = \max\{\lambda \in \Pi_\rho : f \succsim_c [\bar{y}, \lambda : \underline{y}]\}$; and
- $\bar{\lambda}_f = \min\{\lambda \in \Pi_\rho : [\bar{y}, \lambda : \underline{y}] \succsim_c f\}$ - - LUB

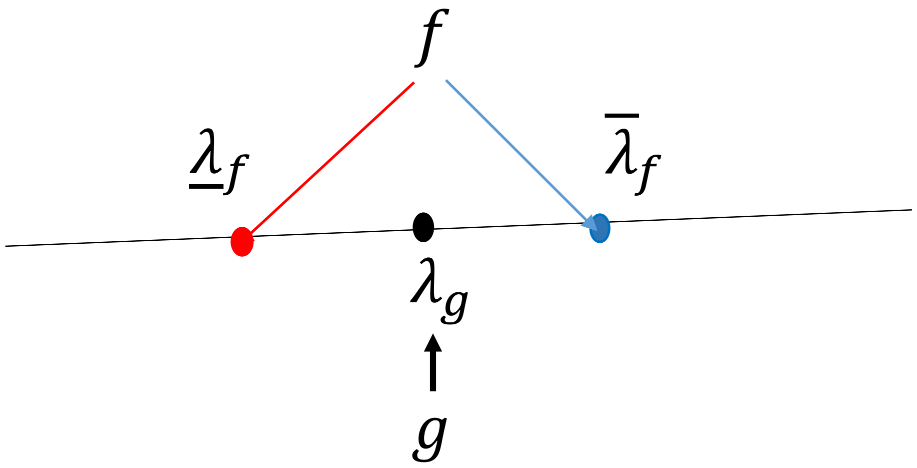
- $\bar{\lambda}_f = \underline{\lambda}_f$ if $f \in M(F;\rho)$; and $\bar{\lambda}_f > \underline{\lambda}_f$ if $f \notin M(F;\rho)$.

Case **IA**: $f \notin M(F;\rho)$ and $g \in M(F;\rho)$ - - only g is measured;

Case **IB**: $f, g \notin M(F;\rho)$ - - neither is measured.

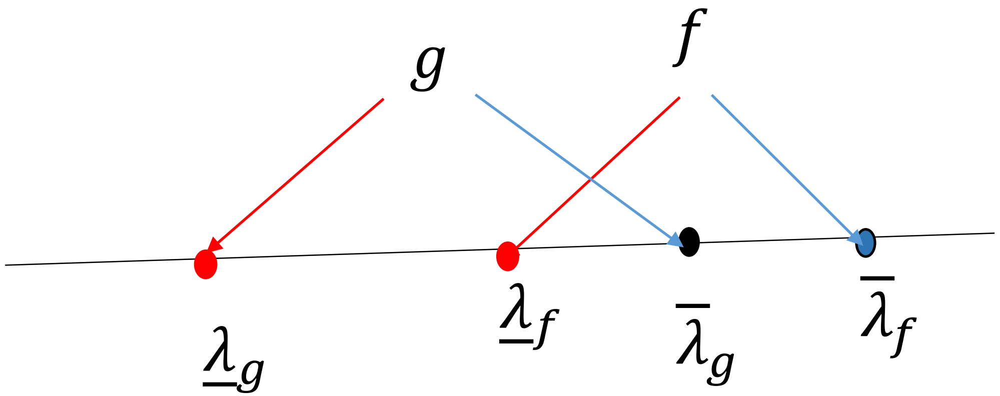
- The incomparability relation by $\bowtie_c$, i.e., $f \bowtie_c g \Leftrightarrow_{def}$ Neither $f \succsim_c g$ nor $g \succsim_c f$.

Theorem IA: $f \bowtie_c g \Leftrightarrow \underline{\lambda}_f < \lambda_g < \overline{\lambda}_f$.



Theorem IB: **(1):** $f \succ_c g \Leftrightarrow \underline{\lambda}_f \geq \overline{\lambda}_g$.

(2): $f \bowtie_c g \Leftrightarrow \underline{\lambda}_f < \overline{\lambda}_g$ and $\underline{\lambda}_g < \overline{\lambda}_f$.



Representation Theorem.

$$f \succsim_c g \Leftrightarrow min\,\{\underline{\lambda}_f, \overline{\lambda}_g\} \geq max\{\underline{\lambda}_f, \overline{\lambda}_g\}.$$

KT example (p.5) with $\rho = 2$

- The leftmost are obtained by letting $a \wedge c = 0\%$;
- The middles are, assumed independence;
- The rightmost are, letting $a \wedge c = 20\%$.

Case A: $a = [\bar{y}, \frac{80}{100}; \underline{y}] \succ_c y = b$.

$\lambda_y < \frac{80}{100}$ - - risk lover.

Case B: $a = [\bar{y}, \frac{80}{100}; \underline{y}] \prec_c y = b$.

$\frac{80}{100} < \lambda_y \leq 1$ - - risk averse (e.g., $\lambda_y = \frac{85}{100}$).

	c: 65%	d: 35%
a:20%	$a \wedge c$: 0 // 13 //20 EU	$a \wedge d$: 20 // 7 //0 A-paradox
b: 80%	$b \wedge c$: 65 // 52 //45 A-paradox	$b \wedge d$: 15 // 28 //35 EU

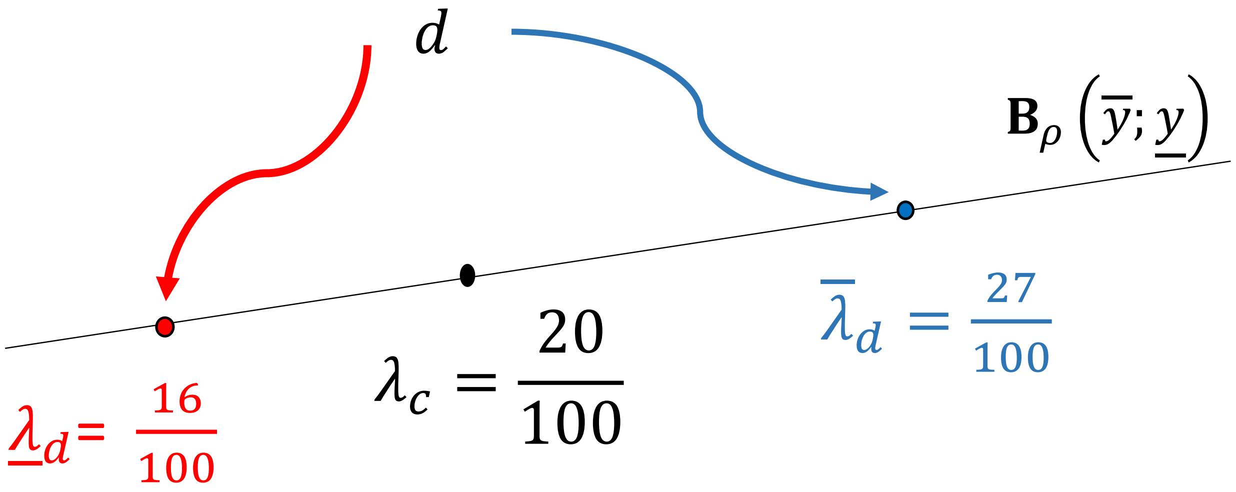
- Recall where $c = [\bar{y}, \frac{20}{100}; \underline{y}]$ and $d = [y, \frac{25}{100}; \underline{y}]$

- d and c are incomparable, since $\underline{\lambda}_d < \lambda_c < \overline{\lambda}_d$.

Interpretation:

- Since d or c are incomparable, either d or c could be chosen, **provided he is asked to choose either.**

	c: 65%	d: 35%
a:20%	$a \wedge c$: 0 // 13 //20 EU	$a \wedge d$: 20 // 7 //0 anomaly
b: 80%	$b \wedge c$: 65 // 52 //45 anomaly	$b \wedge d$: 15 // 28 //35 EU



We say the theory explains the KT result

- $a \succ_c b$ or $c \succ_c d$;
- any preference observed in the experiment is consistent with the theory with the choice of parameter λ_y.

- the theory explains the KT result $\Leftrightarrow$ $\rho = 2$.

Measurement λ_y by **Satisficing/aspiration** due to Simon

Decision Maker asks himself how he evaluates by the thought experiment:

Q_1: for each $\lambda \in \Pi_1$, is y indifferent to $[\overline{y}, \lambda; \underline{y}]$?

- If he finds a satisfactory one, he accepts λ_y as his decision.
  - if there are multiple ones, λ_y is between the lowest λ_y^- and highest λ_y^+.
- Otherwise, he goes to the step 2:

- Step 2 is essentially the same but he considers Π_2 instead of Π_1.
- Goes on until he gets tired.

- Suppose that the satisficing function is single peaked. If the above step stops Π_k with $\lambda_y \in \Pi_k$, then precision of λ_y *is not greater than* $|\lambda_y^+ - \lambda_y^-| \leq \frac{1}{10^{k-1}}$.

Over all, the above argument is “doxastic” decision, rather than looking for an intrinsic existence of “preferences” in the mind.

Conclusions

We developed the EU theory with probability grids.

- It covers the range from classical EU theory to bounded cases.
- In bounded cases, it allows multiplicity of preference relations exhibiting some incomparabilities.
- Utility representation theorem - - an analytic tool.

- We applied the theory to the Allais paradox.

Remaining Problems

- Experimental tests?
  - Incomparabilities are not directly experimentable; how do we ask “comparable or not”?
  - However, the concepts GLB and LUB are experimentable.
- An approach to “subjective probability” may be in the scope of our theory.

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